The Converse of Sarkovskii's Theorem

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In this paper, we will complete Robert Devaney's discussion and proof of Sarkovskii's Theorem and its converse. Sarkovskii's theorem is particularly powerful and interesting because it give us information about the periodicity of the orbits of *any* continuous mapping of the real line. Before we complete the proof of Sarkovskii's Theorem, we must consider the Sarkovskii ordering of the natural numbers, which lists all odd numbers except one, followed by these odds times 2, followed by 2^2 times the odds, and so on until we exhaust all positive integers except the powers of 2, which we list in decreasing order. That is,

$$3 \triangleright 5 \triangleright 7 \triangleright ...2 \cdot 3 \triangleright 2 \cdot 5 \triangleright 2 \cdot 7 \triangleright ...2^m \cdot 3 \triangleright 2^m \cdot 5 \triangleright 2^m \cdot 7 \triangleright ...2^3 \triangleright 2^2 \triangleright 2 \triangleright 1.$$

Theorem 1 (Sarkovskii's Theorem) *Suppose $f : \mathbb{R} \to \mathbb{R}$ is a continuous function, which has a periodic point of prime period k. If $k \triangleright l$ in Sarkovskii's ordering, then f also has a periodic point of period l.*

Proof: As Devaney already proved the cases where $k = 2^m$ and k odd for us, we need only prove the theorem for prime period $k = p(2^m)$ when p is odd.

Suppose f has a point with prime period $k = p(2^m)$ where p is an odd integer. Then f^{2^m} has some point(s) with prime period p. By Devaney's proof for odd k, f^{2^m} also has a point of period $n(2^r)$, where n is odd and $n > p$ or $r \geq 1$ or $n = 1$. Therefore, f also has points of prime period $n(2^{m+r})$, with the same restrictions on r, m, n. Having completed Devaney's proof, we can now move to our main topic, the "converse" of this theorem.

Theorem 2 (Converse of Sarkovskii's Theorem) *For each $k \in \mathbb{Z}^+$, there exists a continuous $f : \mathbb{R} \to \mathbb{R}$ such that for each $l \triangleright k$, f has a point of prime period k but no points of prime period l.*

Proof will be offered in four separate lemmas corresponding to four subsets of the natural numbers which encompass Sarkovskii's ordering.

Lemma 1 *For all $n \in \mathbb{Z}^+$, there is a continuous, $f : I \to I$, from a closed interval to itself such that f has a point with prime period $2n + 1$ but no points of period $2n - 1$.*

Proof: We want to find a function such that (1) $f^{2n+1}(x) = x$ for all $x \in I$ and (2) $f^{2n-1}(x) \neq x$ for some $x \in I$. Define f on $I = [1, 2n+1]$ as follows:

$$f(x) = \begin{cases} nx + 1 & x \in [1, 2] \\ -x + 2n + 3 & x \in [2, n+1] \\ -2x + 3n + 4 & x \in [n+1, n+2] \\ -x + 2n + 2 & x \in [n+2, 2n+1]. \end{cases}$$

After calculating the orbit of the interval 1 under iteration by $f(x)$, we find that seed is periodic with prime period $2n+1$, which satisfies the first condition for this proof. To satisfy the second requirement, we first examine the orbit of $f(x)$ on any closed interval $[j, j+1]$, for $j \in 1, n+1]$. After enough iterations, the orbit is at $[j+1, 2n+1]$, which clearly is not periodic with period $2n-1$. Repeated $2n-1$ iterations of the interval $[n+1, n+2]$ eventually produces $[1, 2n+1]$. So f^{2n-1} has a fixed point, since it spans the entire interval. These fixed points will either be fixed points or periodic points of $f(x)$. When $f^a([n+1, n+2]) \in [n+1, n+2]$ these fixed points satisfy $f(x)$ as fixed points and when $f^a([n+1, n+2]) \notin [n+1, n+2]$ then it is in $[j, j+1]$, which we already discussed.

Lemma 2 *For all $n \in \mathbb{Z}^+$, there is a continuous function $f : I \to I$, which maps a closed interval to itself, such that f has a point with prime period $2^k(2n+1)$ but no points of period $2^k(2n-1)$.*

Proof: Here we want to create a function $F(x)$ that doubles the period of f, the function from Lemma 1, originally and then continues doubling the period with each iteration. We can do this with the following function:

$$D(x) = \begin{cases} f(x) + 2z & x \in [0, z] \\ x - 2z & x \in [z, 3z], \end{cases}$$

where $[0, z]$ is the definitional domain of f above. First, $D(x)$ has no periodic points in $[z, 3z]$. Also, notice that each of the intervals $[0, z]$ and $[z, 3z]$ maps to the other subinterval. That is, $D([0, z]) = [2z, 3z]$ and $D([z, 3z]) = [0, z]$. Moreover, since $[0, z]$ is the original domain of $f(x)$, whenever $x \in [0, z]$, $D^2(x) = f(x)$. So, if a periodic $x \in [0, z]$ has prime period n under iteration of f then x has prime period of $2n$ under iteration by $D(x)$ and if $y \in [0, z]$ is periodic with period $2n$ in $D(y)$ then y is also periodic with period n under iteration of $f(y)$. So, k iterations of $D(x)$ satisfies this lemma.

Lemma 3 *For all $n \in \mathbb{Z}^+$, there is a continuous $f : I \to I$ such that f has a point of period 2^n but not of 2^{n+1}*

Proof: This function can be easily created by repeated iteration of the $D(x)$ from lemma 2 on a new function: $f(x) = x$ for $x \in [0, 1]$, which has periodic points of only period 1.

Lemma 4 *For all $\in \mathbb{Z}^+$, there is a continuous $f : I \to I$ such that f has a point $3(2^n)$ but no points of period $(2m-1)2^{n-1}$.*

Proof: Define a continuous $f : [1,3] \to [1,3]$, which is linear between points on a 3-cycle: $f(1) = 2, f(2) = 3, f(3) = 1$. Doubling f as in Lemma 2 yields a point with prime period 6, satisfying the requirement that f have a periodic point of prime period $3(2^n)$.

We will prove that $D(x)$ has no points of period $(2m-1)2^{n-1}$ by induction. If $n = 1$, then the prime period is odd; however, odd periods are unattainble in D^{n-1} by Lemma 2. That is, $D([1,3]) = [5,7]$ and $D([5,7]) = [1,3]$, as seen above. Now we can suppose that this lemma is true for $n-1$. Then $D^n(x)$, the nth doubling of $f(x)$ has a periodic point of prime period $3(2^n)$. If $D^n(x)$ has a point of prime period $(2m-1)2^{n-1}$ then D^{n-1} must have point of period $(2m-1)2^{n-2}$. However, this period is unattainable to D^{n-1} but our assumptions for induction.

With these four proofs, we need notice only one other fact to complete our proof of the converse of Sarkovskii's theorem. If the constructed function for any of the lemmas has a point of period b and if $b \triangleright c \triangleright d$ then the function also has points of periods c and d, which implies that if no points of prime period c exist then no points of prime period b exist. Thus, the four lemmas cover all possible k in Sarkovskii's order and our discussion of Sarkovskii's Theorem ends.