Solution set 2

Warmups

Warmup problems are quick problems for you to check your understanding; ***don't turn them in.***

1. The half-life $\tau_{1/2}$ of a radioactive substance is the time until only one-half of the substance remains. How is $\tau_{1/2}$ related to the time until $1/e$ of the substance remains?

> Since e is bigger than 2, the $1/e$ time is slightly longer than the half-life:
>
> $$\tau_{1/e} = \tau_{1/2} \ln 2.$$

2. You are a ship navigator back in the old days when clocks were the only way to measure longitude. If your expensively constructed clock has lost 10 minutes after, say, a sea voyage of 1 month, by roughly how many degrees will you be in error about your longitude? More importantly, by roughly how many miles will you be in error about your position? Should you be worried? (Assume that you are at 45° latitude.)

> In 24 hours the earth rotates a complete revolution, which is 360° in longitude. So 10 minutes, which is $1/(24 \times 6)$ of 24 hours, is $360/(24 \times 6)^\circ = 2.5^\circ$.
>
> At the equator, each degree is a distance $2\pi R/360 \sim 70$ miles, where R is the radius of the earth. At 45° latitude, this distance is less by a factor of $\sqrt{2}$, so each degree is about 50 miles. The navigation error is about 120 miles – enough to run aground on unexpected sandbars or rocks.

Problems

Turn in solutions to these problems.

3. Estimate the size (in dollars/year) of the US diaper market. These market-sizing questions are often asked in management-consulting interviews.

> There are roughly 8 million babies in the United States, as estimated in lecture or in Chapter 14 of the notes. Each baby may have its diaper changed every few hours, for say 8 per day (as I found out in June). Each diaper may cost \$0.50, so the daily cost is \$32 million. Per year that becomes
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> $$\frac{\$32\text{ million}}{1\text{ day}} \times \frac{365\text{ days}}{1\text{ year}}.$$
>
> You can do the calculation mentally by using a restricted number system where 1 and 'few' are the only numbers (except in exponents, which can be any integer). So 32 million is a few times 10 million, and 365 days is a few times a hundred. The rule is that few times few equals 10, so

$$32\text{ million} \times 365 \sim \text{few} \cdot 10^7 \times \text{few} \cdot 10^2 = (\text{few})^2 \cdot 10^9,$$

which is 10^{10}. So

$$\frac{\$32\text{ million}}{1\text{ day}} \times \frac{365\text{ days}}{1\text{ year}} \sim \$10^{10}/\text{year}.$$

I don't know the true answer. The few reports that I found indexed online are available for fees varying from \$500 to \$2000. So it is a valuable computation!

4. Estimate the integrals by replacing the smooth curve with a rectangle, using the FWHM (full-width, half-maximum) heuristic (Chapter 15 of the notes) to choose the rectangle. How accurate is each estimate?

a. $\displaystyle\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx$. [Exact answer: π.]

The maximum of $y = 1/(1+x^2)$ is $y = 1$, at $x = 0$. The half-maximum is $y = 1/2$, which is attained when $x = \pm 1$. So the full width at the half maximum (FWHM) is 2. The discretized area (shaded) is therefore $2 \times 1 = 2$. The exact integral is

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = \tan^{-1} x \Big|_{-\infty}^{\infty} = \pi.$$

So the estimate of 2 is low by roughly 40%. Not terrible, but not great.

b. $\displaystyle\int_{-\infty}^{\infty} e^{-x^4}\,dx$. [Exact answer: $\Gamma(1/4)/2 \approx 1.813$.]

The maximum is again $y = 1$ at $x = 0$. The half-maximum of $y = 1/2$ is attained when $x = \pm(\ln 2)^{1/4}$. So the discretized area is $2 \times (\ln 2)^{1/4} \approx 1.825$, an overestimate of 0.6%.

5. The period of a pendulum is approximately

$$T = 2\pi\sqrt{\frac{l}{g}}\left(1 + \frac{\theta_0^2}{16}\right),$$

where θ_0 (measured in radians!) is the angle at which the pendulum is started.

Roughly how many seconds per day does a pendulum clock with $\theta_0 = 10°$ lose compared to a pendulum clock with $\theta_0 = 5°$, if both have the same string length l? [See Problem 2 if you want to know whether the loss of those seconds is significant for navigation.]

The clock with the larger amplitude has a longer period, so it ticks less often in a day than does the clock with the smaller amplitude. How many fewer ticks? Compute it by fractional changes.

The fractional difference in period comes from the fractional change in the factor $1 + \theta_0^2/16$. This factor is roughly 1, and it changes by $\alpha^2/16 - \beta^2/16$, where α is the large amplitude of $10°$ (measured in radians) and β is small amplitude of $5°$ (measured in radians).

Numerically, the fractional change is roughly 0.0014. A day has 86 400 seconds, so that fractional change in period results in a loss of $86\,400 \times 0.0014$ seconds, which is about 100 seconds.

Bonus problems

Bonus problems are more difficult but ***optional*** *problems for those who are curious.*

6. Guess the functional form of $T(\theta_0)$ for $\theta_0 \approx \pi$.

If the pendulum started at exactly $\theta_0 = \pi$, then it would never move, ignoring quantum fluctuations. If it starts close to π, then the deviation from π will grow rapidly in a positive-feedback loop: As the deviation grows, so does the gravitational torque, so the deviation grows even more rapidly, etc. When θ_0 is very, very close to π, most of the period comes from waiting for the deviation to get large.

To estimate the period, we therefore need to know how the deviation grows. The equation of an inverted pendulum is just the same as the regular pendulum but with the opposite sign for the torque. So:

$$\frac{d^2\phi}{dt^2} - \frac{g}{l}\sin\phi = 0,$$

where $\phi = \pi - \theta$ is the deviation from vertical. When ϕ is small, this equation results in exponential growth with time constant $\sqrt{l/g}$. So an initial angle $\pi - \phi$ turns into $\pi - \phi e^{t/\sqrt{l/g}}$ after a time t. Let's say that the exponential-growth phase of the motion stops when ϕ is roughly 1. For ϕ to reach 1 takes a time $\sqrt{l/g}\ln(1/\phi)$, which is our estimate for the period. This estimate assumes that the exponential-growth phase takes much longer than the typical period, an assumption that is true only when the initial angle is very, very, very close to π.

Therefore, in extreme case of an initial angle near π, the period grows logarithmically – i.e. very slowly – to infinity as the initial angle gets closer to π.