[SQUEAKING] [RUSTLING] [CLICKING]

CASEY RODRIGUEZ: All right, so last time we proved the following theorem-- that if I have a convergent series, and this implies that the limit as n goes to infinity of x sub n equals 0. So a natural question is, as a beginning advanced math class, does the converse hold? Is this a two-way street or a one-way street?

So if the individual terms in this series converges to 0, does this imply that the series converges? And I'm sure you answered this question in some form in a previous calculus class. And the answer to this question is no.

So what's the counterexample? It's the so-called harmonic series, which corresponds to our favorite sequence which converges to 0. So we'll state this as a theorem-- the series sum from n equals 1 to infinity of 1 over n does not converge.

So how are we going to prove this theorem? We'll prove this theorem by showing a sequence of partial sums. Some sequence of partial sums for this guy does not converge. So if it were to converge, then the sequence of partial sums converges, and therefore, every subsequence of partial sums converge.

So what's the strategy? We're going to show that there exists a subsequence of partial sums, here s n k-- let's make this m k-- so remember, the partial sums are simply summing up the first-- so the index here is m sub k, so the first m sub k terms, 1 over n, diverges. And this is enough to show that the full series doesn't converge again.

Because if it did converge, the sequence of partial sums converges, and therefore, every subsequence of partial sums converges. So if we're able to show there exists a subsequence which diverges, then we're done. In fact, what we're going to do is something a little bit stronger. We're going to show that there exists a subsequence of partial sums which not only diverges but is unbounded.

And therefore, the entire sequence of partial sums is unbounded. So if we're able to show there exists a subsequence which is unbounded, then the entire sequence of partial sums is unbounded, so it can't converge. Because remember, convergent sequences imply bounded sequences.

So we're going to look at when m sub k is dyadic. And for some reason, I switched indices from k to l in my notes, so instead we're going to go from m sub l. So let l be a natural number, and we'll consider the partial sum corresponding to adding up the first 2 to the l terms.

Now, you may ask, why 2 to the l? Why not 3 to the l? Well, 4 to the l will be a subsequence of that, but 2 to the l is-- you could do 3 to the l, you could do 5 to the l, but 2 to the l is sufficient for our purposes.

So what we're going to do is we're going to take this partial sum and bound it from below by something which is quite large. So first off, all of these partial sums are bounded from below by 0. They're a sum of non-negative terms.

So we write s 2 to the l. This is equal to 1 plus 1/2 plus-- and I'm going to put parentheses around that-- plus 1/3, plus 1/4, plus 1/5, plus 1/6, plus 1/7, plus 1/8. So how I'm grouping these terms is I'm grouping them according to whether the denominator falls between a power of 2 and the next power of 2, and then plus dot, dot, dot, 2 to the l minus 1 plus 1 plus 1 over 2 to the l.

Now I can write this partial sum. So I've grouped terms this way. What is this in terms of precise symbols?

This is 1 plus sum from lambda equals 1 to l, sum from n equals 2 to the lambda minus 1 plus 1 2 to the lambda 1 over n So. These lambdas are now parameterizing what power of 2 I'm at. So when lambda equals 1, I'm at this block. When lambda equals 2, I'm at this block, and lambda equals l, I'm at this block.

And then I'm summing up the terms that have denominator between that power and the next of 2. So I have this, and now I bound that sum from below. Because again, I'm trying to show that this subsequence of partial sums is unbounded.

So I bound it from below. I sum from lambda equals 1 to l, sum from n equals 2 to the lambda minus 1 plus 1, 2 to the lambda. And now for each of these, n is between 2 to the-- so this should be 2 to the lambda. Now for n between these two numbers, 1 over n is always bigger than or equal to when I plug in the biggest bound here, 2 to the lambda.

And now I just have a sum, so over n, but there's no n in this term. So I just add up all the number of terms in a given block. And this is equal to-- so first I have 1 over 2 to the lambda coming from here, and then sum from n equals 2 to the lambda minus 1 plus 1, 2 to the lambda times 1. So again, I'm going a little slow here.

And this is just equal to l 1 over 2 to the lambda times the number of terms I have here, which is 2 to the lambda minus 2 to the lambda minus 1 plus 1, so this minus this plus 1. So this is equal to 1 plus lambda equals 1 to l, 1 over 2 lambda. And so this 1 cancels with this one. And then I have 2 the lambda minus 1, which I can bring out.

And so this is just 1. And I get 1 plus sum from lambda equals 1 to l. And this 2 to the lambda cancels with this 2 to the lambda, and I'm left with just a 1/2. And this is equal to-- now remember, there's no sum here in lambda, so this is just 1 plus l over 2.

So what did we do? We basically showed that each of these blocks is bounded from below by a 1/2. That's this term that we get right here in the end.

And we can see this if we just go through the first three terms which I have written here. So 1/2 is clearly bounded below by a 1/2. 1/3 plus 1/4 is bounded below by 1/4 plus 1/4, because 1/3 is bigger than that. So 1/4 plus 1/4 is a 1/2.

If I look at this next block, that's 1 over 5 is bigger than or equal to 1 over 8. So is 1 over 6. So is 1 over 7.

So this sum is bigger than or equal to 1/8 plus 1/8 plus 1/8 plus 1/8, which equals 1/2, plus, and then so on. So maybe I should have said that before I went into the actual computation. But in the end, we get that this subsequence of partial sums s to the 2l.

So let me just summarize. This is bigger than or equal to l plus 2 over 2. And as l gets very large, this thing gets very large.

So this implies s to the 2l, this subsequence, equals 1 to infinity is unbounded, which implies that the full subsequence, or the full sequence of partial sums, is unbounded. And therefore, the sequence of partial sums does not converge. And therefore, that series does not converge. So we see that the converse does not hold for that question or for that theorem.

I will just make a very passing mention to the fact that there are fields for which that does hold-- so not ordered fields, because again, ordered fields with the least upper bound property have to be R, and therefore, we've just shown that the converse of that theorem does not hold. But in fact, if you look at the so-called p-adic numbers, they do have this property that if the sequence of terms converges to 0, then the series converges. But we will never see p-adic numbers. I just wanted to do a little lip service to that fact-- that there are at least fields of numbers that do have this property.

So we had a theorem about limits of sequences and how they interact with algebraic operations. This naturally implies a theorem about series. So let alpha be in R, and let's suppose we have two convergent series.

Then if I look at the series alpha x n plus y n-- so the terms of my new series are alpha times x sub n plus y sub n-- this is a convergent series. And the sum of this series is equal to what you expect. So the sum of the series alpha x n plus y n is equal to alpha times the sum of x n's plus the sum of y n's. So this theorem follows essentially kind of immediately from what we did for sequences.

Partial sums satisfy-- if I look at sum stopping at m, alpha x n plus y n, now just by the linearity of just adding up finitely many terms-- I'm not going to put something down below because I really don't need to-- this is equal to alpha times the partial sum of x sub n plus the partial sum corresponding to y sub n. And so we're assuming this sequence of partial sums converges and this sequence of partial sums converges. So therefore, this term on the right-hand side converges, which implies the left side converges.

So by the linear properties of limits, namely that the limit of the sum is the sum of the limits. And multiplication just by fixed real numbers commutes with taking limits, so we get that limit as m goes to infinity, so the partial sum corresponding to the new series equals alpha plus-- and that's just alpha times sum x n plus sum y n. And that's the end.

So now, remember we had certain sequences which we could tell whether they converge, a little bit easier than just an arbitrary sequence. A couple of examples of-- at least one example of a sequence we could decide if it converges kind of easily is a monotone increasing sequence. And we showed that a monotone increasing sequence converges if and only if it's bounded.

So we're going to use this to be able to say something about series now-- not sequences, but series-- that have non-negative terms that I'm adding up. Because the partial sums corresponding to a series that has non-negative terms form a monotone increasing sequence. And that's not too hard to show.

So this is the following theorem. So now, we're going to discuss a little bit about sequences or look at sequences which have non-negative terms. So the theorem is the following-- if for all n a natural number, x then is bigger than or equal to 0-- so all these terms are non-negative-- then the series converges if and only if the sequence of partial sums s sub m-- is bounded.

And again, the way we see this is just that when these terms are non-negative, the sequence of partial sums is monotone. So here's the proof. It's quite easy.

So we have for all n a natural number-- make that m-- if I look at s sub m plus 1, this is equal to-- so this is the m plus 1 partial sum-- from m equals 1 to m plus 1, x of n, this is equal to sum from n equals m x n plus x n plus 1. And now the x m plus 1 term is not negative. We're assuming all the terms are non-negative. So this right-hand side-- this is certainly bigger than or equal to sum from n equals 1 to m of x n, and that equals s m.

So just summarizing for all natural numbers m, the s m plus 1 is bigger than or equal to s m. If I just keep adding non-negative things, the partial sums are getting bigger. So the partial sums is a monotone.

So maybe I should've stated this slightly differently just so that you don't think this is part of one of the if and only ifs. I mean, this is the assumption that we have for this whole statement. Suppose this, so the conclusion is that this converges if and only if the sequence of partial sums is bounded.

So based on the assumption that all the terms are non-negative, we see that the sequence of partial sums is monotone increasing. That's why what we proved for sequences-- every monotone increasing sequence converges if and only if it's bounded. And that's it.

Now, not every series we look at does have non-negative terms. But we can always form a certain series from those terms to make a new series with non-negative terms, which gives us information about the original series. What am I going on about? And look at the convergence properties of that new series.

So we have the following definition-- that a series converges absolutely, or we say we have absolute convergence, if the series formed by taking the absolute values of these terms, if this series converges. So what I was trying to get at before I stated this definition is that absolute convergence implies usual convergence. If I had this series converging absolutely, then the original series converges.

Now, before I prove this theorem, let me prove a little, small theorem. I can't remember if I gave it for an assignment or not, but it's essentially a triangle inequality for however many terms you like. So we'll prove this theorem in just a minute.

But first, let me prove the following theorem-- that if m is bigger than or equal to 2, and x1 up to x m are in R, then sum from n equals 1 to m x n, take the absolute value of this sum, this is less than or equal to sum from n equals 1 to m of the absolute value. When m equals 2, this is just the usual form of the triangle inequality. So m equals 2-- this is just saying x1 plus x2 is always less than or equal to x1 plus x2, which is just the triangle inequality.

But typically how life works, at least in analysis, if you can do it for two things, then you can do it for n things or m things, in this case, by induction. And so that's how we're going to prove this. So we'll prove first prove this triangle inequality by induction.

Now, in the induction proofs we've done so far, little n is our thing that we're inducting on. In this statement, m is the thing, induction on m. So let's look at the base case, which is m equals 2.

So then this is just the triangle inequality for two real numbers that we've already proved before-- x1 plus x2 in absolute value is always less than or equal to sum of x1 plus the sum of x2-- I mean, the sum of absolute value of x1 and the absolute value of x2. So the base case is fine. So now we do the inductive step.

So I'll assume the statement that I want to prove. Usually, I use m, but now I'll go to the next letter l, going in reverse alphabetical order. Suppose if x1, x l in R.

So let's actually, instead of just restating all that, just-- I'll just [INAUDIBLE] star. So suppose star holds for m equals l. And now we want to prove that star holds for m equals l plus 1. Now we want to show star holds for m equals l plus 1.

Let x1 up to x l plus 1 be in R. Then if I look at the sum from n equals 1 to l plus 1 x n, this is equal to sum from n equals 1 to l x n plus x l plus 1 in absolute value. By the usual triangle inequality for two terms, this is less than or equal to sum from n equals 1 to l of x n in absolute value plus the absolute value of x l plus 1 by usual triangle inequality.

And now this term, since I'm assuming m equals l holds, so the m equals l case says this is less than or equal to sum from n equals 1 to l of x n plus x l plus 1 in absolute value. So this is by inductive hypothesis. And this is just equal to sum from n equals 1 to l plus 1 x n. so we've proven the case for now m equals l plus 1. And that concludes the proof of this generalized triangle inequality with arbitrary number of terms.

So let's get back to proving this theorem, that absolute convergence implies convergence. So we'll do that by proving that absolute convergence implies that the series is Cauchy. So proof-- and this is of the theorem just before this theorem, we proved that absolute convergence implies usual convergence.

So we will prove that in fact, this series is Cauchy, assuming absolute convergence. And from last time, we had approved the statement, or at least this followed from the statement for sequences, that a Cauchy series converges-- that a series is Cauchy if and only if it converges. So we have to prove that the series is Cauchy.

Remember, this means for all epsilon positive, there exists a natural number m such that for all l bigger than m bigger than or equal to M, if I look at the sum from n equals m plus 1 to l of x n, this is less than epsilon. So let epsilon be positive. So since we're assuming that the series is absolutely convergent, this implies that this series with absolute values here is also Cauchy.

So that means that there exists a natural number m sub 0 such that for all L bigger than m bigger than or equal to M sub 0, if I look at the sum of absolute values from m plus 1 to l, this is less than epsilon. Now, this should have an absolute value on the outside, but this is a sum of non-negative terms, so the absolute value can be removed. You can essentially see where we're going based on what's written on the board-- what we want to prove, and what we know, and this triangle inequality.

So choose M to be M sub 0. Then if l is bigger than m is bigger than or equal to M, then the absolute value of the sum m plus 1 to l sub n-- this is less than or equal to the sum from n equals m plus 1 to l of the absolute values of x sub n by the theorem we proved just a minute ago. And this is less than epsilon by our choice of M. M is equal to M0, and for M0, we have this inequality right here. Thus the series is Cauchy, which implies it converges.

Basically, the only test you know for determining when a series is convergent is in one of two possibilities. Either A, it has a very simple form, and so all the terms are non-negative, but the terms have a very simple form. It's the alternating series test which we'll discuss in a little bit, possibly the next lecture.

And then when a series converges absolutely, we have a lot of tests for that. And we'll see that series which converge absolutely somehow are not fickle, meaning I can rearrange the terms and the rearranged series will still converge absolutely, and converge to the same thing that the original series converged to.

So let me just make a brief comment after this theorem, we proved that absolute convergence implies usual convergence, and tie-in a little bit to what I just said there. So we'll show that the series sum from 1 to infinity of minus 1 to the n 1 over n-- this converges. But note that this series does not converge absolutely. Because when I take absolute values, I just get sum of 1 over n, which is the harmonic series, which we just showed a few minutes ago is divergent.

So now we're going to move on to some convergence tests. Now, when it comes to convergence tests, what these all follow from is basically what we know about geometric series and the following comparison test, although when I do the proofs of the other convergence tests, I won't actually state that I'm using the comparison test. But that's kind of what's really getting used.

So the first test we have is the comparison test. And the statement is the following-- suppose for all n a natural number, we're looking at non-negative terms with one smaller than the other. Then the conclusion is if the bigger one converges, this implies that the smaller one converges. And if the smaller one diverges, this implies that the bigger one diverges.

How we're going to prove this is-- so we're dealing with terms that are non-negative, so we'll use this theorem about a series of non-negative terms. So we use that theorem, and we proved that a series of non-negative terms converges if and only if the sequence of partial sums is bounded. So if this series converges, this implies that the sequence of partial sums is bounded, which implies-- that means that there exists a non-negative number such that for all natural numbers m, sum from n equals 1 to m of y sub n is less than or equal to B.

But this immediately implies that since all the x n's are less than or equal to the y n's, we get that the n-th partial sum corresponding to the x n's, which is less than or equal to the n-th partial sum for the y n's, is also less than or equal to B for all m. So sequence of partial sums corresponding to the x n's is bounded. And therefore, by the theorem we proved, which I think I have erased already, the series converges.

Now, proving 2 is essentially-- it's kind of the same thing, except the inequalities go the other way. And the x n's are getting bigger, which implies the y n's are also getting-- or the partial sums corresponding to the x n's is getting bigger implying that the partial sums corresponding to the y n's are also getting bigger. So now 2, if this series diverges, then this implies that partial sums is unbounded. we'll now prove that this implies that the partial sums corresponding to the y n's are unbounded.

Now, remember what it means for a sequence to be bounded is that there exists a non-negative number B such that for all m I have that bound. So to say it's unbounded means that for all B there exists a little m bigger than or equal to capital M such that that inequality is reversed. So let me put here in a box what this actually means.

Again, this means that for all B bigger than or equal to 0, there exists m, a natural number, so that y n equals 1 to m is bigger than or equal to B. That's what this means. So this is a for all statement, so I have to be able to prove it for every B, that B be bigger than or equal to 0.

Now, since we know that the partial sums corresponding to the x n's is unbounded, this implies that there exists an m, a natural number, such that the sum from n equals 1 to m of x n is bigger than or equal to B. So let me say again, to show its unbounded I should really have-- so for a sequence to be unbounded, I should have an absolute value here is bigger than or equal to B. But all of these terms are non-negative, and therefore, I can remove the absolute values, and the same thing here.

So there exists a natural number m so that I have this. And so we put a little m0 there because we have to somehow show there exists a little m. Choose m to be this m0.

So now, if we look at the partial sums for the y n's, this is bigger than or equal to-- because m equals m0, it's this term, which is bigger than or equal to B. Thus, this proves the partial sums corresponding to the y n is unbounded. And therefore, this series diverges.

So let's use the comparison theorem to consider series like 1 over n p-series, and prove when they do converge. So theorem-- for p, a real number, sum n equals 1 to infinity 1 over n to the p converges if and only if p is bigger than 1. So for the proof, why does the series converging imply p has to be bigger than 1?

I'll do this by contradiction. So suppose 1 over n to the p equals 1 to infinity converges. So we'll do the proof by contradiction that p has to be bigger than 1.

Suppose p is less than or equal to 1. Then 1 over n to the p-- where p is less than or equal to 1, this is bigger than or equal to 1 over n. And this implies, since 1 over n-- since the series corresponding to 1 over n-- diverges implies that the series corresponding to 1 over n to the p diverges by the comparison test, which is a direct contradiction to what we're assuming, that the series converges.

So this must be false. p must be bigger than 1. So we've shown that if this series converges, then p has to be bigger than 1.

So now let's prove the other direction and suppose p is bigger than 1, and prove that the p series, 1 over n to the p, converges. So the way we're going to do this is kind of how we showed that the harmonic series is divergent. So what we're going to do is first, we're going to show that there is a subsequence of partial sums corresponding to this guy that is bounded.

So remember, to prove that this converges, this converges if and only if the sequence of partial sums is bounded. And what we're going to first do towards that is prove that there is a subsequence of partial sums which is bounded. So we make a first claim that the sequence of partial sums-- so s 2 to the k, this is sum from n goes 1 to 2 to the k of 1 over n to the p, so k a natural number-- this partial sum is bounded by a fixed number depending on p, 1 plus 1 minus 2 to the minus p minus 1.

In other words, this subsequence of partial sums corresponding to s 2 to the k is bounded. So again, we do this by grouping these terms according to which power of 2 the denominator is between, and then estimate from above now, rather than from below like we did for the harmonic series. So we have s 2 to the k equals 1.

So again, we're grouping these terms according to where they fall. So just write this out one more time-- this is equal to 1 plus 1/2 to the p plus 1 over 3 to the p plus 1 over 4 to the p plus 1 over 5 to the p plus 1 over 8 to the plus-- and then up until the last term. And now I can write this as 1 plus the sum from l equals 1 to k, so the number of blocks I have here. and now, the terms that come in each of these blocks 1 over into the p.

And so now I estimate 1 over n not from below by this guy, but from above by putting in the smallest n that n is in this block. So this is less than or equal to 1 plus sum from l equals 1 to k sum from n equals 2 to the l minus 1 plus 1, 2 to the l 1 over 2 to the l minus 1 plus 1 raised to the p-th power. Now this plus 1 is just making things bigger on the bottom, so if I remove it, I've made things bigger overall for this fraction.

So this is less than or equal to sum from 1 equals l equals 1 to k, sum n equals 2 to the l minus 1, 2 the l times 1 over 2 to the p times l minus 1. And now this thing here, if we do the same algebra we did a minute ago, this is equal to 1 l equals 1 to k. Now I have this term coming out.

And then the number of terms I have here, just like I did for the harmonic series, this is equal to 2 to the l minus 2 to the l minus 1 plus 1 plus 1. And now this is equal to 1 plus sum from l equals 1 to k. And so this whole thing here is equal to 2 to the l minus 1.

So I get 2 to the minus p minus 1 l minus 1. Now I can shift this index. Actually, I guess I could have made that sharper but it doesn't matter. I could shift this index l by-- no.

So l starts at 1 and goes to k. And here, I have the sum l minus 1. So I can shift this index to go from now l equals 0 to k minus 1, 2 to the minus p minus 1 l. So this is like making a change of variables, l prime equals l minus 1. And so let me put l prime instead of l.

So p is bigger than 1, so this corresponds to a geometric series now. So let me actually rewrite this as 1 over 2 to the p minus 1 to the l prime. When p is bigger than 1, then 1 over 2 to the p minus 1 is less than 1.

So this thing is a k minus 1 partial sum for the geometric series with this as R. So this is always bounded above by-- if I add up all the terms, which equals that thing that I have up there, 1 over 1 minus 2 to the minus p minus 1. So that proves that along this subsequence, these partial sums are bounded by this fixed number.

And now I claim that this proves that the whole sequence of partial sums is bounded, in fact, by the same number. For all m, a natural number, s m is less than or equal to this number again-- 1 minus 2 to the minus p minus 1. So let m be a natural number, so we're trying to prove this bound.

What do we do? We find a dyadic number, a number of the form 2 to the k bigger than m. And since 2 to the m is bigger than m-- I think that's maybe one of the first things we did by induction-- we get that s sub m-- which is the partial sum of non-negative terms-- this is going to be less than or equal to, since this is a monotone increasing guy, this is going to be less than or equal s to the 2m, which is less than or equal to 1 plus 1 minus 2 to the minus p minus 1. Thus, the sequence of partial sums is bounded, which implies this series converges. And that's the end of the proof, and I think we'll stop there.