[SQUEAKING]

[RUSTLING]

[CLICKING]

CASEY RODRIQUEZ: So we're going to continue with our discussion of the derivative. So now, let me recall the definition we introduced at the end of last time of the derivative. So let I be an interval, meaning it could be open, closed, it could go out to plus infinity, it could go out to minus infinity. But you know what an interval is.

And let's take a function from that interval to R. So we say, f is differentiable at a point c in I. If this limit of f of x minus f of c over x minus c, this difference quotient, if this limit exists. If this limit exists, we also denote it by f prime of c.

And last time, we showed that-- we gave a simple example, that from last time, that if x equals alpha times x to the n, then this function is differentiable at every c, and its derivative is equal to n times alpha x c to the n minus 1.

Now, let's state this in an equivalent way using sequences. Remember, we had this characterization of limits of functions. And the function we're looking at is f of x minus f of c over x minus c. We had this equivalence between limits and limits of sequences from last time, that the limit as, let's say, g of x as x goes to c equals L, if and only if for every sequence x sub n converging to c g if x sub n converges to L as n goes to infinity.

So we can restate what it means for a function to be differentiable at a point and its derivative to be L, say, if and only if for every sequence x sub n, with x sub n not equal to c for all n, and converging to c, we have that L is equal to this limit as n goes to infinity of now this sequence of numbers, f of xn minus f of c over x of n minus c.

So today, the theme of today is the connection between differentiability and continuity. And we have a very easy implication, which is that if a function is differentiable at a point, then it must be continuous at that point. So that's a statement of this theorem, if f going from an interval to R is differential at c, then f is also continuous at c.

So let me also add in here that continuity at the point c in an interval is equivalent to saying that the limit as x goes to c of f of x equals f of c. Now, this is just a subtle point that I want to make, is that for an interval, open, closed, half open, half closed, whatever, any point in that interval is a cluster point of that interval.

And therefore, this definition I made up there is actually meaningful. And for continuity, something being continuous at c is equivalent to-- if c is a cluster point, which I just said it is-- it always is, this limit equals the function evaluated at the point. So that's just kind of a subtle comment I want to make.

So how do we prove this? We write the limit as x goes to the c of f of x as in this way. So essentially, what I do is, I add and subtract f of c. And then, the part that is f of x minus f of c, I multiply by x minus c and divide by x minus c, which is perfectly fine. Because remember, for a limit, I'm never actually looking at points where x equals c.

So that's fine. And so, I get this expression here. Now, what we know about limits, is that the limit of the sum is the sum of the limits as long as all these limits exists. And the limit of the product is a product of the limits, again, assuming all of these limits exists. And the limit as x goes to c of everything you see here do exist. So as x goes to c, this thing in brackets, or in the-- I guess these are brackets. I don't know what those are called.

This limit here is f prime of c. As x goes to c of x minus c, that's just 0. And f of c is just a constant. There's nothing changing with x. So the limit of a constant is that constant. So I just get f of c. So the limit of this whole expression is f prime of c times 0 plus f of c, which equals f of c. So we've just proven that the limit as x goes to c of f of x is equal to f of c, which is what we wish to prove. And therefore, a function which is differentiable at a point must be continuous at that point.

So as beginning math students, we're learning proofs, but we're also learning questions, what types of questions to ask. And whenever you come across a new theorem that has one implication, one hypothesis, one conclusion, then you should ask yourself, does the converse hold? Does the conclusion also imply the hypothesis?

So we've shown that f differentiable at c implies f is continuous at c. So does the converse hold? Namely, if f is continuous at a point, does this imply that f is differentiable at that point? And as you can see already there, the answer is, no. I think you've probably covered this in calculus. But what's the function that gives this counterexample that is continuous at c but not differentiable at c?

Well, let's take, for example, c equals 0. And let's look at the function f of x equals the absolute value of x. In this function here, this is continuous at every point. So in particular, at c equals 0. But it's not differentiable at c equals 0. And how do you come up with-- and how do you do that? How do you show something's not differentiable? The simplest way is to use this remark up here by finding a sequence converging to c, so that the difference quotient does not have a limit as n goes to infinity.

So that's what we'll do here is, we'll find a sequence x sub n, with x sub n not equal to 0 for all n converging to 0, such that this limit does not exist. Rather than write that, because the limit does not exist, it doesn't make really sense for me to even write that. Let me write this as saying, as the sequence is divergent.

Now, how are we going to come up with this sequence, x of n, so that the difference quotient does not converge? Well, what's the logic? So for f of x equals the absolute value of x, it looks like this. It looks like x-- it's equal to x over here minus x over here.

So if you were to formally differentiate-- well, I mean, you could actually differentiate to the left of 0, you get the derivative is equal to minus 1. To the right, the derivative is 1. So that kind of suggests maybe there's some funny business going on at 0. So let's look at a sequence which alternates between being negative and positive, but it's converging to 0, and see if that sequence will provide us with this desired sequence which results in the difference quotient being divergent.

So let's just take a guess. Let xn be minus 1 to the n over n. It doesn't have to be this one, just something that alternates back and forth would be enough. So this could be n squared here, or 2n, or n to the 2020, whatever you like. Then, this is clearly always non-zero. And it converges to 0.

And of course, why it converges to 0? I mean, you could prove this by epsilon m definition, but also if I take the absolute value, that's less than or equal to 1 over n. And this one we know converges to 0, 0. So by squeeze theorem, the thing in the middle must converge to 0. And we're done.

So we have this sequence x sub n equals minus 1 to the n over n, which is alternating back and forth. And we hope it's going to result in this difference quotient being divergent as n goes to infinity. And so, we compute that. Let's look at the difference quotient f of x sub n minus f of 0. F of 0, which is 0. So I just get f of x sub n, which is the absolute value of minus 1 to the n over n over x sub n, which is minus 1 to the n over n.

This is just equal to-- so this is just equal to 1 over n over minus 1 to the n over n, which equals minus 1 to the minus n. But minus 1 to the minus n, this is equal to the same thing as minus 1 to the n.

So this sequence here of difference quotients, so f of xn minus f of 0 over xn minus 0, this is just equal to the sequence minus 1 to the n, which we know is divergent, doesn't have a limit. That's one of the first examples of a sequence we proved does not converge. So in summary, this function f of x equals the absolute value of x is continuous at c equals 0, but not differentiable at c equals 0.

So now, a natural question that people asked is, so a function being continuous at a point does not necessarily imply that it's differentiable. But let's take a function which is continuous on the real number line. Is there a point where it's differentiable? So it's already on the board. But this is not completely a crazy question to ask.

For example, let's go back to f of x equals the absolute value of x. This function is differentiable everywhere except 0. So there's lots of points where this function is differentiable. It's differentiable for c positive and c negative. So for this continuous function, there does exist points where it's differentiable. And you can imagine trying to draw any kind of curve that you can on a piece of paper.

And you can probably find a point on that curve that has a tangent. I mean, if you sit there with a pencil and then try to draw something very jagged, I mean, there are still going to be little sections of your jagged curve where it has a tangent. So I imagine this is why people thought this was the case, that a continuous function has to have at least one point where it's differentiable.

And Weierstrass, the godfather of analysis, said, no. He came up with a whole class of examples of functions, which are continuous on the real number line, but are differentiable nowhere. And this was a really surprising set of examples and result to the community.

And to me, it's also one of the few results in this class that you really didn't see in your calculus class. So we're going to go through this example. Although in most analysis classes, it's reserved for later. But I think we can do it now.

And so. What we're going to prove is-- so we're going to construct a function which is continuous everywhere but is differential nowhere, so a continuous nowhere differentiable function. And I'll even write down the function for you.

So what's the idea that Weierstrass had? It was, let me-- well, I'm not going to write down the function just yet. But the idea of the constructing such a function is that it should be highly oscillatory. Imagine, again, you're trying to draw a picture of the graph of a function that's nowhere differentiable.

You would sit there and break your pencil trying to draw just a highly oscillating function so that it never has a tangent. And that's his idea, is to build a function which is highly oscillatory but too oscillatory so that it's still continuous at every point.

So to start with the construction of this function, which is continuous but nowhere differentiable, we're going to need a few simple facts to start off with. So again, let me write out-- state our goal. We're going to construct a continuous function from R to R, which is nowhere differentiable Not differentiable, it's differentiable at no point.

So let's start off with a few simple facts about-- so I said that we're going to build a function which is oscillating quite a lot. There's two functions you know that oscillate cosine and sine. So those are going to be our building blocks. We'll choose one of them. Let's choose cosine, say, as to be our building block.

But anyway, so let's start off with some elementary facts about cosine. So first is that for all x, y, and R, cosine x minus cosine y is less than or equal to the absolute value of x times x minus y.

The second is that for every real number c, and for all k natural number, there exist a y in c plus pi over k, c plus 3 pi over k, such that-- and let me label this as theorem one-- such that cosine of kc-- that's funny-- minus cosine of ky is greater than or equal to 1.

So both of these are simple facts about cosine. Really, this one just follows from the angle sum formula. This one follows from the periodicity of cosine. But so, why are these true? So first off, we did prove-- and maybe I should have used sine instead of cosine in all of this.

But if you recall from our continuity section, we proved that for all x and y, sine x minus sine y is less than or equal to the absolute value of x minus y for all x and y. And there was a simple relation between cosine x and sine of something. And that's just a shift by pi over 2.

Then cosine of x minus cosine of y, this is equal to sine x plus pi over 2 minus sine y plus pi over 2. And if you like, let me instead of using x and y here, let me use a and b. So then, this is less than or equal to x plus pi over 2 minus y plus pi over 2 using this inequality. So that proves number one.

For number two, it's just a simple fact about cosine being 2 pi periodic. So function g of x equals cosine k of x. This is also periodic. But now, what's the period? It's 2 pi over k.

Now, this interval has link 2 pi, except it's missing the point c plus pi over 2 and c plus 3 pi over k. Thus, if I look at the image of-- OK.

Then, so this function g of x equals cosine k of x. If I look at the image of this interval, this will contain all of the real numbers between minus 1 and 1. So this is a period. This is a interval of link 2 pi over k. Cosine kx is 2 pi over k periodic.

So I should get everything between minus 1 and one except possibly the value at the endpoints. So take away-- and the value at the endpoints is minus cosine ck. Because cosine of k times c plus pi is equal-- pi over k is equal to cosine of kc plus pi. And cosine of something plus pi is equal to minus cosine of that thing.

So the image of the set by cosine kc contains everything between minus 1 and 1, except for possibly the endpoint. This really only happens at if cosine of k times E equals 1. So OK. Then, if cosine of kc is bigger than or equal to 0, then we choose y in this interval c plus pi over kc plus 3 pi over k.

Let's write it this way. So if this thing is bigger than or equal to 0, then we choose y-- why am I confusing myself? So this is simple enough. We choose y in this interval so that cosine of ky equals minus 1. And if cosine kc is less than 0, then we choose y so that cosine of ky equals 1.

So again, here's-- so forget-- just to get an idea. So I'm quibbling over a minor point, the fact that we have to take away possibly the point where-- possibly the value of cosine kc. Anyways, that's erase this for a minute. And let's imagine that the range of this guy-- I mean, it's a 2 pi over k periodic interval.

So this should basically cover all values between minus 1 and 1. So cosine of kc will either be plus or minus. So if it's not negative, then I can find a y in this interval, since it's 2 pi over k periodic. So that cosine of ky equals minus 1. And since this is not negative and this is equal to minus 1, the difference between these two in absolute value will be greater than or equal to 1.

Now, if cosine of kc is less than 0, then again, since this interval contains minus 1 to 1, it's 2 pi over k periodic, we choose a y so that cosine of ky equals 1. And then, this is negative. This is 1. So the difference between something negative and 1 is greater than or equal to 1. And that's the proof.

I made a kind of a mess of it. But this is very easy to understand if you just draw a picture really. So let's continue. And again, what these two parts of this theorem say is that cosine can be quite oscillatory if you insert a k here.

Because the difference between this interval is actually quite small. The length of this interval is 2 pi over k. If k is very large, that's a very small interval. Yet somehow, we can find two points which differ by at least 1 in value if we plug them into the function.

But cosine is not too wild, because it satisfies this bound. So these two are kind of-- they're going to be the ingredients in that idea I said at the start, that we're going to build a function which oscillates, which is quite oscillatory, but it's not too oscillatory that it's still continuous. And we're going to build our function out of cosine kx, where k is going to be changing.

So I'm going to need one more very simple fact. So this one, I won't mess up too much. Which is the following. For all a, b, c, and R, the absolute value of a plus b plus c, this is bigger than or equal to the absolute value of a, minus the absolute value of b, minus the absolute value of c.

And the proof of this is just to use the triangle inequality twice. We have the absolute value of a. This is equal to a plus b plus c, minus b plus c. And this is less than or equal to, by the triangle inequality, the absolute value of a, plus b, plus c, plus the absolute value of minus times b plus c. But that minus goes away with the absolute value.

And then, I use a triangle inequality one more time here to get-- and then if I subtract-- oh, I didn't want to do that. And so, if I subtract this side over to the other side of this inequality, I get the statement of the theorem.

So now, we're going to introduce-- call this theorem two-- now let me introduce the guest of honor. So first, I have the following claim. For all x in R, the series given by sum from n equals 0 to infinity-- and instead of n, I'm going to use k, cosine 160 kx. So here's our very oscillatory guy over 4 to the k is absolutely convergent.

So for each x in R, this series converges absolutely. So it spits out a real number. So I can define a function in terms of this series. Let f from R to R be defined by a prime-- f of x is just the number that I get when I stick in x to the series. Which is meaningful, because for all x, this series converges absolutely. So this is just a function. I put in x. I get out a real number. Then the claim is, f is bounded and continuous.

So this is going to be our function, which is continuous but nowhere differentiable. As you can see, it's built out of a bunch of very oscillatory functions, cosine of 160 kx, this is just a really big number here. And as I said, when we were talking about this theorem here, each one of those pieces is very oscillatory. On a very small interval, it oscillates between two values that differ by at least 1.

And so, all of these guys are oscillatory. And they're oscillatory on smaller and smaller intervals. And somehow, we're adding them all up in a way to get a function that's oscillatory on arbitrarily small intervals. And therefore, it will not be differentiable.

So the proof of one is very easy. We just use the comparison principle. Cosine of 160 k times x over 4 to the k, this is less than or equal to-- because cosine of no matter what you plug in is bounded by 1. This is always bounded by 1 over 4 to the k.

And therefore, by the comparison principle, we get that this series is convergent. And therefore, the original series is absolutely convergent. So each of these is bounded by 1 over 4 to the k. This series converges. This is a geometric series. Therefore, by the comparison principle, this series converges.

So now, we have this function defined by this series. Note, this is not a power series, because a power series involves polynomials in x. This is cosine of x, or a number times x. So let's show it's bounded. The same proof actually that we gave here shows it's bounded. Let x be in R then, f of x equals-- this is equal to the limit as n goes to infinity of this sum.

And now, for our absolute values, this limit pulls out. Whenever the limit exists, the limit of the absolute value equals the absolute value of the limit. And this is less than or equal to the limit as n goes to infinity of bringing the [INAUDIBLE]. So by the triangle inequality, this is just a finite sum. So I can bring the absolute values in.

And I get sum from a equals 0 to m of cosine 160 k, x over 4 to the k, absolute value. And this is less than or equal to limit as n goes to infinity of sum k equals 0 to m of 4 to the minus k, again, because cosine of anything is bounded by 1. And this equals 4/3. This is just a sum from k equals 0 to infinity 4 to the minus k. So this function is always bounded by 4/3.

So the function is bounded. Let's now show it's continuous. So to show a function's continuous, remember, we have that other characterization of continuity that a function is continuous at a point if and only if for every sequence converging to that point f of xn converges to c.

So before I start writing all this down, let c be in R, and let xn be a sequence converging to c. So what we want to show is that limit as n goes to infinity of f of xn minus f of c in absolute value equals 0. That's the same as saying the limit as n goes to infinity of f of xn equals f of c.

Now, f as a bounded function. So this sequence, f of xn minus f of c in absolute is a bounded sequence. So it has a lim sup. And we did an exercise on the assignment that the limit of a sequence equals 0 if and only if the lim sup equals 0. So equivalently, we'll show that lim sup of f of x sub n minus f of c equals 0.

This thing always exists for a bounded sequence. Which is one of the reasons which is what makes lim sup so useful, is that they do always exist. So if we show that this lim sup equals 0, then this is equivalent to showing this limit equals 0.

Again, this was an exercise, where you can take it as the lim inf of something that's non-negative is always bigger than or equal to 0. So if we show this is equal to 0, we would have 0 is less than or equal to the lim inf of this thing, is less than or equal to the lim sup, which equals 0. And therefore, the lim inf equals the lim sup equals 0. So that's another way of saying that this is equivalent to this.

So we're going to show this at the lim sup of f of x sub n minus f of c equals 0. But we don't-- so that might be a little tough. But what we can show, and what we will show, is we'll give ourself a little room. We'll show that for all epsilon positive, the lim sup of f of x sub n, minus f of c, which is a non-negative number, is less than epsilon.

So this is a fixed number, non-negative number, which is always smaller than-- put less than or equal to there-- which is smaller than any number that I want. And therefore, it has to be 0. And proving thus, lim sup.

So just to recap, we want to show-- we have a sequence converging to c. We want to show that f of xn converges to f of c. Another way of stating that is that this limit of the absolute value between-- of the difference between these two converges to 0. That's again a equivalent way of stating the limit.

And in another assignment, we proved that for a sequence of non-negative numbers, or if you like, just for a sequence of numbers, it converges to 0 if and only if the lim sup of the absolute value converges to 0. But this is a sequence of non-negative numbers, this absolute value of f of xn minus f of c. So this is equivalent to this. So this thing we want to show is equivalent to this thing right below it.

Now, rather than show directly that this limit lim sup equals 0, we're going to show that for all epsilon positive it's less than or equal to that small number. And therefore, proving that it's 0, because it's a non-negative number smaller than every positive number. So this is our goal, to show this lim sup is less than epsilon.

So let epsilon be positive. We're now going to prove that that lim sup is less than epsilon. So first off, let m0 be a natural number such that the sum from k equals m0 plus 1 to infinity of 4 to the minus k is less than epsilon over 2. So this series here, right this is a convergent series if I go from k equals 0 to infinity.

And we have this Cauchy criterion for convergent series, which can be equivalently stated as for all m0 and natural number, there exists-- or for all epsilon, there exist-- so first off, rather than go through all that, we can actually just compute this. That the left-hand side equals 4 to the minus m0 minus 1, times sum from L equals 0 to infinity, 4 to the minus L, which equals 4/3. 4 to the minus m0 not minus 1, which equals one 1/2, 4 to the minus m0.

So if m0 is chosen very large-- this is 4 to the minus m0. So this is-- I can write this as 1/2 over 1/4. And as long as m0 is chosen very large, I can always make this very small. That's the left side over there. So I can always find m0 so that this is the case.

And now, we compute the lim sup of f of xn minus f of x. We split this up into two pieces. This is equal to lim sup then of two parts, sum from k equals 0 to m0 of cosine of 160 k xn minus-- so each of these is defined in terms of a sum.

So I'm going to break the sum up to m0. And then, everything past m0, this is equal to 1 over 4k times cosine 160 k xn minus cosine 160 kx, plus sum from k equals 0 to m0 plus 1 to infinity, 1 over 4 k, same thing, absolute value.

And now, we use the triangle inequality and the fact that lim sups preserve inequality. So the absolute value of this thing is less than or equal to the sum of the absolute values. And the lim sup of those sequences is less than or equal to the sum of the lim sups. That was another exercise from an assignment, or you did something similar with the lim int.

So this is less than or equal to lim sup of-- now I use the triangle inequality, sum from k equals 0 to m0, 4 minus k, cosine 160 k x of n minus cosine 160 k-- oh dear. So I've been writing x. I meant to write c. Sorry about that. c, c, c, c. OK. And then, same thing in these brackets here. Now, [INAUDIBLE] might keep that.

So we have that this lim sup of f of x sub n minus f of c is less than or equal to the lim sup of this guy, plus the lim sup of this guy. And now, I'm going to use the triangle inequality, again, bringing the absolute values inside of the sums, which is perfectly valid, even for the infinite sum by the same argument I used basically over there.

This is less than or equal to the lim sup as n goes to infinity of minus cosine of 160 kc, and sup plus lim sup of sum from k equals m0 plus 1 to infinity of, now, the absolute value of the same thing. Now, I applied the triangle inequality to that.

Now, cosine of anything is always bounded by 1. So this is bounded by-- first off, let me come back to this second one. m0 is fixed. Remember, this is not changing within. It's just fixed. It depended only on epsilon. So m0 is fixed. And this is less than or equal to lim sup of int. Now we use what we know about cosine, that cosine of something minus cosine of something else is bounded by the difference in the argument.

So this is bounded by minus 4 to the minus k times 160 k xn minus 160 kc, plus lim soup of k equals m0 plus 1 to infinity, 4 minus k times 2. As this is bounded by 1, that's bounded by 1.

Now, there's no n left here. So this is really just now equal to that. And we chose m0 so that this quantity here is less than epsilon over 2. So times 2, this is less than epsilon. So this is less than or equal to lim sup, sum from k equals 0 to m0 of 40 k.

Now, again, this is just a fixed number. The lim sup is in n. This is just a fixed number times x sub n minus c. The difference in this is just equal to 160 k times 4 to the minus k, that's 40 k. And then, we chose m0 so that this quantity over here is less than epsilon.

And now, as n goes to infinity, x sub n is converging to c. So this thing here converges to 0, equals times 0 plus epsilon equals epsilon. So to me, this is a real first proof of analysis, where you're using everything that you've been exposed to up to this point to prove a really deep theorem. So go through this slowly.

But there's not too much-- the estimates are not all that tricky once you have them in front of you. So we've shown that for all epsilon, this lim sup is less than or equal to epsilon. Because remember, we ended with epsilon here. But we started off estimating this lim sup here.

Now we're in a position to prove the final theorem that we want to prove. And this is due to Weierstrass. The function which we've been studying, f of x equals sum from k equals 0 to infinity cosine 160 kx over 4 to the k is nowhere differentiable.

So with everything we've done so far, so we this function is continuous. We've proven that. So this theorem provides you with an example of a function which is continuous but nowhere differentiable. And with what we have on the board, namely the key parts are going to be what's in that first theorem there. And with this triangle inequality we have over here, we'll be able to prove this.

So in fact, let me give myself a little space. And I'm going to state theorem two from over there, because I need some space to right in a minute. For all a, b, c, the absolute value of a, plus b, plus c is bigger than or equal to a minus the absolute value of b minus the absolute value of c.

So let me re-summarize. Since I've summarized it already. What we've done up to this point, we've shown that this function here is well-defined. Absolutely, the series is always absolutely convergent. And therefore, this function is bounded, and/or we also proved that it's bounded and continuous. That was the previous theorem.

So this function is bounded and continuous. And we're going to prove it's nowhere differentiable. And again, what's the idea? The idea is that we've built it out of functions which are highly oscillatory at smaller and smaller scales.

So somehow this function is highly oscillatory at every scale. And if you have a function which is highly oscillatory of every scale, if some of you have heard of Brownian motion, which is a function which is-- I mean, which is a path which is highly oscillatory, then that function will not be differentiable anywhere.

So proof, let c be any real number. And what we're going to do, just as in when we looked at f of x equals the absolute value of x, we're going to construct or find a sequence x sub n, such that x sub n does not equal c for all n, xn converges to c.

And the sequence, f of x of n, minus f of c, over x sub n minus c, in fact, is divergent. But we'll go even further, is unbounded. And therefore, the sequence cannot converge. And therefore the function is not differentiable at c.

So we're going to use theorem one to find the sequence. By theorem one, namely part two, for all N, a natural number, there exists the x of n, such that what? x sub n is in this interval of the form. So c plus pi over 160 n is less than x, is less than c plus plus 3 pi over 160 n.

And cosine of 160 n x of n minus cosine of 160 nc is bigger than or equal to 1.

So let me call these two properties a and b. So another way of writing a is, we could have instead written, by subtracting c across the board, that x sub n minus c is between pi over 160 n and 3 pi over 160 n. So by a, for all n, x sub n does not equal to c, because their difference is bounded below by a positive number.

I mean, their difference is actually positive, and it's non-zero. So-- and by the squeeze theorem, again, if you like putting the c on both sides, we get that x sub n converges to c. So this will be our sequence-- our bad sequence, for which the difference quotient is unbounded.

So in fact, let me write this out a little bit more. And x sub n minus c, which is equal to x sub n minus c, because this is positive, is less than 3 pi over 160 n. And this goes to 0. Therefore, the absolute value of xn minus c converges to 0. And therefore, x sub n converges to c.

Now, to lessen the number of times I have to write cosine 160 k, let fk of x be cosine 160 kx over 4 to the k. So f of x is equal to the sum of k equal to 0 of fk of x.

And now, what we're going to do is, we're going to find a lower bound on the absolute value of f of xn minus f of c over x sub n minus c. So now, find a lower bound on-- if I can find a lower bound on this absolute value, which is getting large, as large as I wish, then that proves that this sequence is unbounded, and I'm done.

So let's look at the absolute value of f of xn minus f of c. Let's just write this in a few different ways. Or not a few different ways. But let's split it up as a sum. So all of these guys are equal to a sum of f sub k. So I have the n'th one. So remember, f sub k, this is just one of these building blocks. Plus sum from k equals 0 to n minus 1, plus I put x here. I meant to write c.

And so, I'm going to let a sub n be this first number, b sub n be the second number, including the sum. And this will be c sub n. So this is equal to a sub n plus b sub n plus c sub n.

And now I'm, going to use that triangle inequality I proved. Then f of x sub n minus f of c is greater than-- which is equal to a sub n plus c sub n plus c sub n is bigger than or equal to a sub n, minus b sub n, minus c sub n.

Now, a sub n, this is going to be-- this is just the difference between these guys. This is bounded below by 1. So it's kind of large. Not 1, but 1 over 4 to the n. These other guys we'll prove are very small compared to the a sub n. And then, this lower bound-- this upper bound we have on x sub n minus c will be the nail in the coffin, as we say.

So our goal is now to estimate from below a sub n-- remember, we're trying to find a lower bound for this quantity over x sub n minus c. So we need to estimate these things from below, or this sum from below. That means, we need to estimate this from below, and then b sub n and c sub n, since they have minus signs, from above.

Now, by b, a sub n, which is equal to 4 to minus n, cosine 160 nx of n, minus cosine 160 to the nc. This is-- remember how we chose these x sub n's. This is bounded above by 1, or bounded below by 1. So this is bigger than or equal to 4 to the minus n. So that's a sub n.

Now, let's look at how big b sub n is. Now, we want to bound this from above. Because when it hits the minus sign, this will flip the inequality, and we'll have that this would be bounded from below by something. So the absolute value of B7, that's equal to sum from k equals 0 to n minus 1 of fk of x of n minus f of c, k of c.

And bringing the absolute values inside by the triangle inequality, this is less than or equal to sum from k equals n minus 1 of f of fk x of n minus fk of c. And now, so this is equal to sum from k equals 0 to n minus 1, 4 to the minus k, cosine 160 k, x sub n minus cosine of 160 kc.

And now we use theorem one, number two-- or number one. I'm sorry, theorem one part one. So this is theorem 1, 1. The difference in these is bounded by 160 k times the difference in those two.

So that's-- and now, these things we can sum up in closed form. Well, actually, x sub n minus c, we proved is less than 3 pi over 160 n. So this is less than a sum from k equals 0, n minus 140 k. And this equals 3 pi over 160 n.

Now, sum from k equals 0 to n minus 1 40k, we used this formula for summing a geometric sum. This is equal to-- what was this equal to? 40 to the n minus 1 over 39. And which is less than-- OK, so take away the one.

This is 1 over 13, four to the minus n times pi. And that's less than-- pi is less than 4. So plus 1. So let me summarize this.

It is less than or to 1 over 13 fours and minus n plus 1. Now, for the last box, this is less-- the absolute value of c sub n. And absolute value is less than or equal to-- now, we just kind of do a brutal estimate.

We bring the absolute values inside. So this is equal-- so just as we had before, with the b sub n's, except now we're summing from k equals n plus 1 to infinity. We brought the absolute values inside.

And then, let's use the triangle inequality on that. This is-- remember, fk is equal to cosine of 160 k times x over 4 to the k. So that's no matter what you plug-in, that's bounded by 1 over 4 to the k.

And now this we can actually sum. This is equal to 4 to the minus n minus 1, times sum from L equals 0 to infinity of 4 to the minus L. What does this equal? This equals 2 times 4 to the minus int minus 1 times 4/3 equals 2/3 4 to the minus n. So, my board work's getting a little shoddy, but it'll be OK. I.e. we've proven that the absolute value of c sub n is bounded by 2/3 4 to the minus n.

So combining everything that we've done. We've shown that f of x sub n minus f of c. Which remember, is bounded below by a sub n-- remember, so the first box is bounded by 4 to the minus n. So first off, everything, the a sub n's, b sub n's, c sub n's, they have some forward to the minus n involved.

What happened to my estimate for b? Oh, covering it up-- 4 to the minus n. So this is one coming from the a sub n's, and minus 4 over 13 coming from the b sub n's, minus 2/3 coming from the c sub n's. And if you do the arithmetic, this is 4 to the minus n, 1 over 39.

So f of x sub n minus c is bounded from below by 4 to the minus n times 1 over 39. Now I just divide by x sub n minus c, which is bounded above by 3 pi over 160 n. And therefore, when I take reciprocals, I get that.

I get that the absolute value of f of x of n minus f of c over x of n minus c is bounded from above by 1 over x of n minus C times 4 to the minus n, 1 over 39, which is bounded below by now inserting this estimate here. I get 40 n over-- I think that's 1017 pi. Yes.

And therefore, the absolute value of the difference quotient is bounded below by 40 to the n over some fixed number. And this thing on the right-hand side is unbounded, as in, I guess, as in varies because from 1 to infinity. And therefore, the absolute value of this difference quotient is unbounded, which finishes the proof.

So we've done some things, maybe a few things that you haven't seen in calculus course. Definitely haven't seen this, most likely. And really, this is the first result that involves a lot of math that we've covered up to this point to prove a very non-trivial and deep theorem that differentiability is a bit of a miracle. It's a miracle when it happens. OK, we'll stop there. .