[SQUEAKING]

[RUSTLING]

[CLICKING]

CASEY RODRIGUEZ: OK, so let's continue our study of sequences of real numbers. So we've seen special types of sequences, monotone sequences, before. And then in the previous lecture, we looked at sequences obtained from sequences, namely the sequences that give you the lim sup and lim inf. And we showed that these are actually limits of subsequences.

Now I'm going to define what looks like a new class of sequences. But we'll see it's actually not. These are called Cauchy sequences. So "coe-shee"-- not "couch-ee," not "cawt-shee"-- Cauchy, so a French guy. So it's pronounced Cauchy and probably not even pronounced like that. It's probably got a different pronunciation by people who actually speak French.

So what is the definition of a Cauchy sequence? A Cauchy sequence, intuitively, it's a sequence so that if you go far enough out in the sequence, any two entries in that sequence are close together. So convergent sequences had the property that if you go far enough out, the entries in the sequence are getting close to a real number. Cauchy sequence is that any two entries are close to each other.

So a Cauchy sequence, so we say a sequence is Cauchy if we're all epsilon positive, there exists an M, natural number, such that if n is bigger than or equal to M and k is bigger than or equal to M, then xn minus xk is less than epsilon. So maybe not write if. I mean, it's the same statement, but since it looks-- so it'll look a little more like previous statements when we put a "for all" there.

So you have a definition here. It's the definition of a new thing. You should now try to look at an example and then possibly negate the definition to see if you really understand it.

So an example of a Cauchy sequence is x of n equals 1 over n, our favorite sequence. So let's prove this. So all we have is the definition. So we have to verify that x of n equals 1 over n verifies the definition of being Cauchy.

So just like when we try to prove something is convergent, which is a "for all" epsilon statement, the first thing you have to do is let epsilon be positive. And then I have to choose M and show that that capital M produces this statement here.

So choose M, a natural number, so that 1 over M is less than epsilon over 2. So I could phrase that as capital M being bigger than 2 over epsilon, but I'm going to phrase it this way. Now we have to show that it works-- namely, if I take n bigger than or equal to capital M and k bigger than or equal to a capital M, then this difference is less than epsilon.

Then if n is bigger than or equal to M, k is bigger than or equal to M, and I look at 1 over n minus 1 over k, this is less than or equal to, by the triangle inequality, the absolute value of each of these added together, which is just 1 over n plus 1 over k. And since these are both bigger than or equal to M, each 1 is less than or equal to 1 over M 1 over M, so I get 2 over M, which, by our choice of M, is less than epsilon. So x of n equals 1 over n is an example of a Cauchy sequence.

So let's negate the definition, and then we'll look at an example of a sequence which is not Cauchy. And as you'll probably guess, if this is our favorite sequence, which converges, our favorite sequence which doesn't converge will be an example of a sequence which is not Cauchy. And this should shouldn't come as a surprise. Because, again, a sequence which is Cauchy, if you go far enough out, any two entries are close to each other.

But if we look at, for example, the sequence minus 1 to the n, which is just minus 1 plus 1 minus 1, any two entries will differ by-- or you can always choose two entries-- that differ by 2 in distance.

So let's negate this definition to get what it means for something not to be Cauchy. So we'll not write all that out. So x of n is not Cauchy if-- so every time we see a "for all," it becomes "there exists."

If there exists a bad epsilon 0 positive such that for all M, a natural number, you can find two entries further out than M that are greater than epsilon 0 distance to each other. So there exists n bigger than or equal to M and k bigger than or equal to M such that x of n minus x of k is bigger than or equal to this bad epsilon. OK

Again, the definition of Cauchy means that, as long as I go far enough out in the sequence, this distance is supposed to be less than epsilon. So for all epsilon positive, there exists a capital M so that I have this picture. If I choose x sub k plus 1, then it should also be within distance epsilon to x sub n or x sub k.

So they're getting closer and closer together. The negation means that they're not getting closer and closer together to each other. So there exists some small distance so that you can always go as far out as you want and find two entries that are greater than epsilon 0 distance to each other.

So what's an example of that? Like I said, the sequence minus 1 to the n is not Cauchy. That just doesn't look right. There we go. Now it is.

So this is not Cauchy. So that means there should exist some bad epsilon 0. So I can go as far out as I want and find two entries in the sequence differing from each other by epsilon 0 in distance. So basically, I can always find two entries in the sequence which differ from each other by 2. So that'll be my bad epsilon 0.

So if you like, here's a proof. Choose epsilon 0 equals 2. Let M be a natural number. So now we have to find an element of entries in the sequence further out than M whose distance to each other is bigger than or equal to 2.

We can just take M plus 1 and capital M. Choose n equals M and k equals M plus 1. So these are both bigger than or equal to M. Then minus 1 to the n minus 1 to the k, this is equal to 1 minus minus 1 after I factor out a minus 1 to the capital M, which equals 2. So minus 1 to the n is not Cauchy.

So I, at the beginning, said that this will look like a definition of a new type of sequence, but it's not, really. So as it turns out, the elements of the sequence are getting closer and closer together as you go far enough out. So

They're all kind of clustering near each other, which kind of makes you think they're all clustering near something in the real number line. And therefore, maybe, the sequence is convergent. Now, this is true-- and we'll prove this-- that a sequence is Cauchy if and only if it's convergent.

Now, this is true only for the real numbers. And I'll say a little bit about this in a minute-- or not only true for the real numbers, but it's not true for the rational numbers. And I'll get to that in just a second. So what we're going to prove is that a sequence is Cauchy if and only if it is convergent.

So the first thing I want to show is that Cauchy sequences are bounded. So the proof of this statement is essentially the same as the proof that convergent sequences are bounded. So let me draw a picture that goes along with this proof.

So as long as I go far enough out, there exists an M so that for all n bigger than or equal to capital M, I can say this. Let's look at this entry x of M. Then for all n bigger than or equal to capital M, all of the other entries have to be within a certain distance to x sub n, based on the definition of Cauchy.

So let's say I make that distance 1. And let's say 0's over here, just for this picture. So then for all n for all n bigger than or equal to capital M, x sub n lies in this interval here. And therefore, we'll get that x of n is bounded. So the way this picture looks, I'm going to write it this way. It's 1 plus 1.

Now, that handles all n bigger than or equal to capital M. So we just need to deal with the first capital M minus 1 other guy. So maybe there's capital M minus 1 is over here. Capital X sub 1 is over there. Capital X sub 2 is over here. So then our bound will just be this one, which handles all of the n bigger than or equal to M plus the absolute values of these guys that we missed.

So since xn is Cauchy, there exists and M, a natural number, such that for all n bigger than or equal to M, and k bigger than or equal to M, x sub n minus x sub k is less than 1 in distance. So this is certainly true for k equals capital M. So this implies for all n bigger than or equal to M, x sub n minus x sub capital M is less than 1.

So now, if I use the triangle inequality, I can show that the previous implies that for all n bigger than or equal to M, if I look at the absolute value of x sub n, this is equal to x sub n minus x sub capital M plus capital M. And this is less than or equal to the absolute value of this guy plus the absolute value of this guy. And this is bounded by 1.

So in summary, I've shown that for all n bigger than or equal to this fixed integer, capital M, x sub n is less than or equal to x sub capital M in absolute value plus 1. So that's for all little n bigger than or equal to capital M. So now I just need to pick a big enough number that bounds the first capital M entries that are not covered by this inequality. Capital M is fixed.

So let B be the absolute value of x sub 1 plus absolute value of x sub 2 plus this fixed number now. Then for all n bigger than or equal to capital M, I have, by this inequality up here-- this is a sum of non-negative numbers, so this number is certainly bigger than or equal to just this part.

And if I have n bigger than equal to 1 and less than M, then x of n, the absolute value of this guy is going to be one of these that appears here, which is certainly less than or equal to if I add on this number and the others, which is less than or equal to B. So now I've found a B which is non-negative which bounds all the absolute values. And therefore, this proves that the sequence is bounded.

So we've shown that a Cauchy sequence is bounded. And so what I'm now going to show is the following. So again, all of the entries are getting close to each other. They're kind of clustering near each other. So it kind of feels like they want to converge.

And this next theorem says that, well, if you've identified a limit along a subsequence, then, in fact, the entire sequence converges. So of course, this is not true for an arbitrary sequence. If a subsequence converges-- or I should say, for an arbitrary sequence, it's not true that a subsequence converging implies a full sequence converging. We have minus 1 to the n for which a subsequence converges, but the whole sequence does not converge.

But if we make the additional hypothesis that the sequence is Cauchy, then the sequence converges if and only if that subsequence converges. So the statement of the theorem is following. If x sub n is Cauchy and there exists a subsequence which is converging to some number-- call it x-- then the whole sequence converges to x.

So what I was saying right before I stated this theorem is that if I hide this part and just say, there exists a subsequence which is converging to x, this does not imply that the full sequence converges to x. Because we had this example of minus 1 to the n. But if I also assume the sequence is Cauchy, then it does follow that Cauchy plus subsequence converging implies the full sequence converges.

So I want to show that xn converges to x. So want to show-- and we're going to do this just by using the definition, by verifying this through the definition, not using the squeeze theorem or anything like that.

So let epsilon be positive. Since xn is Cauchy, there exist M0, a natural number such that for all n bigger than or equal to M0 and k bigger than or equal to M0, x sub n minus x sub k is less than epsilon over 2. Why this epsilon over 2? Or why should you not be surprised?

Well, we have two assumptions here. So like we did when we did convergence of products of sequences and so on, which had two assumptions, namely two sequences converged to something, typically, that means we'll have two integers coming. We'll choose a bigger integer and then some inequalities to get an epsilon. So that's a little bit of a rambling answer to why we get an epsilon over 2 here, or why we put one there.

Since the subsequence-- so this subsequence converges to x-- there exists another integer, M sub 1 such that if k is bigger than or equal to M sub 1, then x sub n sub k minus x is less than epsilon over 2. So maybe I should have used a different letter here. Let's use a little m. Because I don't want you to think these have to be the same k.

So now we'll choose an integer bigger than both M1 and M2 and show that it works. Choose M to be M0 plus M1. Now we need to show this works. And if n is bigger than or equal to M, so let me, actually, make a first observation before I go to the n bigger than or equal to capital M.

Then, since n sub k is bigger than or equal to k for all of k, a natural number-- just because the n sub k is there in increasing sequence of integers, which starts at least at 1-- and since n sub k is bigger than or equal to k for all k, this implies that the integer n sub capital M is bigger than or equal to M, which, remember, is M0 plus M1, which implies that n sub M is bigger than or equal to M0 and n sub M is bigger than or equal to M1. So I just wanted to make this preliminary observation. And now we'll go to showing that this capital M works.

So now, if n is bigger than or equal to capital M, and I look at x sub n minus x, an absolute value, and add and subtract x sub n capital M capital M minus x and use the triangle inequality, then-- so since n is bigger than or equal to capital M, which is bigger than or equal to M0, that means n is bigger than or equal to M0. And then n sub M we just showed is bigger than or equal to 0. So by this inequality, I get that the first term is less than epsilon over 2.

And now, so M is certainly bigger than or equal to M sub 1. And therefore, I will get that this part is less than epsilon over 2 because of this inequality. So that choice of capital M works. And now, we'll prove the following, that a sequence is convergent if and only if it's Cauchy.

So this is a two-way street. So we need to show the left implies the right and then the right implies the left. So this direction is, in fact, easy. Based on what we've done-- I shouldn't say it's easy-- but what we've done so far, it quickly follows.

So we're assuming x sub n is Cauchy. I'm trying to show it's convergent. So if x sub n is Cauchy, this implies that x sub n is bounded, the sequence is bounded, which implies by the Bolzano-Weierstrass theorem that x sub n has a convergent subsequence. And by the theorem we just proved, if a Cauchy sequence has a convergent subsequence, it must be convergent.

Now, for the converse direction, that xn is convergent implies xn is Cauchy, well, so this should not come as a surprise. Let me draw a picture. Let's suppose x sub n is converging to x, and epsilon is positive. Then, since the xn's are converging to x, if I draw a little interval around x of total length epsilon-- so x minus epsilon over 2 and x plus epsilon over 2-- then I will find, as long as so then there exists M so that, for all n bigger than or equal to capital M, all of the x sub n's lie in this interval.

They all lie in this interval because they have to be within distance epsilon over 2 to x if the xn's are converging to x. And since they lie in this interval, the distance between any two of them can only be as big as the length of the interval, which is epsilon.

So this is essentially the picture of why a convergence sequence has to be Cauchy. So now let's turn this picture into math. We have to verify xn is Cauchy through the definition. That's all we have.

So let epsilon be positive. Since the xn's converge to x, there exists an integer M sub 0, a natural number, such that for all n bigger than or equal to M sub 0, x sub n minus x is less than epsilon over 2.

And so we'll choose the M for our definition of Cauchy to be this M sub 0. And if n is bigger than or equal to M and k is bigger than or equal to M and I look at the absolute value of x sub n minus x sub k and add and subtract x and use the triangle inequality, this is less than or equal to x sub n minus x, an absolute value, plus x minus x sub k.

Each of these is less than epsilon over 2 since n is bigger than or equal to capital M, and k is bigger than or equal to capital M. So this is less than epsilon over 2 plus epsilon over 2 equals epsilon. And therefore, xn is Cauchy.

Now I want to make a brief remark about the previous theorem. So remember how this whole story started off? There was something wrong with the rational numbers, namely, they didn't contain the square root of 2. So we couldn't solve the algebraic equation x squared minus 2 equals 0.

But this also, then, turned into the rationals not being complete in the sense of order. Not every non-empty bounded set had a supremum. It didn't have the least upper bound property.

But you can also interpret this lack of having square root of 2 as somehow saying that the rationals are incomplete in this sense. So hopefully, at the end of this class, we'll be able to get to metrics basis. But so what do I mean by that?

Let's say I look at this statement now within the universe of rational numbers. So now, if this sequence is rational numbers-- meaning sequences are only sequences of rational numbers, limits are only elements of the rational numbers, epsilon is only a rational number, and so on-- then we still have many of the same theorems that we proved-- not all of them, and I'll indicate which ones don't hold.

But if we only work in rationals, then we always do have a convergence implies Cauchy, meaning convergent sequences are Cauchy. But Cauchy sequences are not necessarily convergent. Again, what's the example here, or what's the intuitive example?

Take x sub n so that x of n is in Q. And now viewed in the universe of real numbers, x sub n's converge to root 2. Then such a sequence would be a Cauchy sequence. We just proved that, basically.

So such a sequence would be a Cauchy sequence of rational numbers. However, it would not converge in the set of rational numbers. It would converge to the square root of 2, which is not a rational number. So because the square root of 2 is not a rational number, this shows that the rational numbers do not have this completeness property that Cauchy sequences converge.

So there's a whole, still, to this day, kind of industry of studying spaces for which Cauchy is equivalent to convergent. These are called complete metric spaces. And then if you add a little more structure, they're called Banach spaces and so on, which are very important, not just in math but also for formulating rigorously a lot of the underlying assumptions for mathematical physics.

So if we're just looking inside the rationals, it does not follow that Cauchy sequences always converge. And now let's just stop for a minute and take stock of why this was true for the real numbers. What did we use going back?

So if you really go back to the proof of the Bolzano-Weierstrass-- so that's what we used here to show that Cauchy sequences converge-- we use the fact that the lim sup and the lim inf always exists. And lim sup and lim inf, first off, they're defined to be sups and infs, which may not always exist as rational numbers, as we've already shown. So that's definitely a problem already there.

But even more so, when we prove that every bounded monotone sequence converges, what we showed was that this limit is actually a sup of a certain set or an inf of a certain set, which, again, may or may not exist if we're just looking in the rational numbers. Because the rational numbers do not have the least upper bound property.

So it really is the least upper bound property that gives us convergence equivalent to Cauchy for the real numbers. So for R, the least upper bound property is-- it has to be because that's the main thing that separates the two fields, but I'm just reiterating this here-- is the reason why convergent is equivalent to Cauchy.

Now that I've proved that Cauchy is equivalent to convergence, maybe you'll ask, then why did we introduce it at all? If these two notions are the same, why even introduce them if they're just convergent sequences already?

And the reason is because to show that a sequence converges, you have to somehow have your hands on a candidate for the limit. If you want to prove that xn converges, you have to somehow come up with an x that it converges to. And it's not always clear how to find that x.

But Cauchy, although it's equivalent to a convergent in the set of real numbers, doesn't require you to find a candidate for convergence. All it requires you to do is show that, as long as you go far enough out, any two entries in the sequence are close together without requiring you to come up with a limit.

See, computing limits is quite difficult. We're about to do series. And there's maybe, I don't know, five series people can compute explicitly. But you do know that there's a ton of other series that are actually convergent, even though you don't know what the limit as.

And why do you know that? This is exactly because and exactly why people thought of Cauchy sequences to begin with in much of analysis.

So again, just to summarize this, convergent sequences are nice. But in practice, it's difficult to get your hands on what could be a limit of a sequence, especially if that sequence is pretty complicated.

So if you're trying to show a certain sequence converges, it suffices, by what we've done here, to show that it's Cauchy. And that's a little bit easier to do because that just requires you to work with the original sequence. You don't have to come up with a limit. You can just take your sequence and start playing directly with the entries rather than try to come up with a limit explicitly.

So that's what we're going to move on to is series now, which, as I said a minute ago, is original reason why people started developing the foundations of analysis, what we're talking about right now, to begin with. Because they were just kind of doing very formal things that ended up not making sense, like they were adding infinitely many positive numbers and coming up with a negative number.

Well, that can't be right. So all of this was created, discovered-- however you want to phrase it-- to put on rigorous foundations this next topic, which is series.

And you dealt with series in calculus, so you know what a series is. Maybe you don't remember all the proofs of the properties of series. But suffice it to say, series is a pretty good motivation since it's one of the most useful things that comes out of math.

Series expansions are how you solve ODEs, PDEs, Taylor expansions. All these things are, in some sense, a form of series. So being able to justify them as being real things is a necessity.

So the definition of a series, for now, really is just this symbol I'm about to write down. So given a sequence x sub n, the symbol-- or maybe I'll just sometimes write just the sum x sub n-- is what's called the series associated to the sequence x sub n. So right now, that's just a symbol.

We're going to interpret this as a real number in the following situation. We say that the series converges, if the following sequence given by s sub m equals-- so s sub m, this is the element of the sequence.

And what is it? It is the actual sum. So this is not a formal thing. This is just a finite sum from n equals 1 to m. So this is-- so this sequence now, m equals 1 to infinity. And these guys we call partial sums converges.

So right now, if we just have a sequence x sub n, the series associated to that sequence is just a symbol. We say that this series converges if this sequence of partial sums converge and if s is this limit. And we write s is equal to the series and treat the series now as a number.

So in general, if I have a sequence, I just have this formal symbol, which I'm writing down, which I call a series associated to it. In the case that the sequence of partial sums converges, then I actually identify this series with a real number and treat it as a real number.

And so the way I've written this, the series is starting at 1. But it doesn't necessarily have to. So just by shifting the index-- so let me just say here that we don't necessarily have to start a series at n equals 1.

So this could be sum from n equals 0 to infinity, in which case, we have a sequence starting now at x sub 0. Or this could be starting at 2, in which place the sequence of x sub n starts at n equals 2. And then the sequence of partial sums would start at not m equals 1 but m equals 0 or m equals 2, depending on where the series starts.

So some examples. The series sum from n equals 1 to infinity 1 over n plus 1 times n, this is a convergent series. So why is this? So let's look at the proof.

So we look at the m-th partial sum-- this is the sum from n equals 1 to m of 1 over n plus 1n. And this is equal to-- now, if I write 1 over n plus 1 times n as 1 over n minus 1 over n plus 1, this is now the sum of 1 over n plus-- these are finite sums, so I can always split them up. This should be a minus.

And so now, this is equal to 1 plus 1/2 plus 1 over m minus 1/2 plus 1/3 plus 1 over m plus 1 over n plus 1. And you see all of these cancel. And all that's left is 1 minus 1 over m plus 1.

So the m-th partial sum is equal to 1 minus 1 over m plus 1. And therefore, the sequence of partial sums is the limit as m goes to infinity of this, which is just 1. And therefore, this series converges.

Now, our favorite sequence, which does not converge, will give us a series which does not converge. So let's look at sum from n equals 1 to infinity minus 1 to the n. This does not converge. So what's the proof?

The m-th partial sum, this is equal to minus 1 plus 1 plus minus 1 up until I get minus 1 to the m. And therefore, this is always equal to one of two things. If m is odd, then I have an odd number of these guys.

And therefore, the minuses and pluses cancel, just leaving a minus 1 in the end-- this last one, the odd one. And m is odd and 0 if m is even. If I add up an even number of these terms, then all of the minus 1's and 1's cancel out, so I just get 0.

And therefore, this sequence, which is just minus 1 for m odd, 0 for m even, does not converge. And therefore, the series does not converge. So when I write this, this is just a symbol. This is just chalk on a chalkboard. It doesn't mean anything.

So let's go to another series, which does converge. And this is kind of the one to which we compare all other series, essentially, as you'll see. You have all these series tests that you remember, hopefully, from calculus that tell you when a series converges.

But maybe, if you remember the proof or don't, how you do that is you converge it to one series, which you do know how to sum. So it was just by pure luck we were able to compute the sum, or compute the explicit series for this guy.

Another one which we can do that for is geometric series. So the theorem is if I have a real number with absolute value less than 1, then the series starting now at 0 R to the n converges. And I can actually compute the sum of this series. And this is 1 over 1 minus r.

So what's the proof? Let's look at the partial sums. And we can actually compute these, as well, just as we were able to do for the first example. We compute that the sum from n equals 0 to m of r to the m-- now, you can prove this by induction.

I cannot exactly remember if I did this-- I believe I did-- in the second lecture on induction, first or second lecture on the induction. But if I add up some number raised to the n-th-- so this should be to the n-- power from 0 to m, this is equal to 1 minus r to the m plus 1 over 1 minus r. And I guess two lectures ago, we proved that if the absolute value-- so let me state this now.

Two lectures ago, we proved that if the absolute value of r is less than 1, then limit as, let's make it m, of r to the m equals 0, which implies that-- so this was the m-th partial sum-- which implies that the limit as m goes to infinity of s sub m equals the limit as m goes to infinity of this thing, which is, if you like-- so that plus 1 there just multiplies r to the m by r. And using the algebraic facts we proved about limits is 1 minus r time 0 over 1 minus r, which equals 1 over 1 minus r.

So now you ask about the other. What about r bigger than 1? Well, when r equals minus 1, then we get the second example we looked at. If we get r equals 1, then that's just summing up 1, and I'll leave it to you to check that that does not converge, that the sequence of partial sums, if I just sum up 1 from n equals 0 to m, is equal 2n plus 1, which does not converge.

And let me make just a kind of silly comment. And maybe I didn't explicitly make this comment about sequences. Maybe I forgot to do that, as well. So for a sequence, you could start your sequence not necessarily at the first entry.

Maybe you look at a, if you like, new sequence where-- well, so we know from sequences that subsequences of convergent sequences converge. So if I, instead of looking at the whole sequence, start at, let's say, x100 and then go x101, x102, and that's the sequence I look at, well, that's a subsequence of the original one, which, if it converges, implies that subsequence converges.

So all that is to say-- and for this simple way of obtaining this new sequence-- all of that is to say that to understand if a sequence converges, I don't have to consider what happens for the first finitely many terms in the sequence, meaning a sequence x sub n converges if and only if a sequence starting now at say, n equals 100-- and 101, 102, 103 and so on-- converges. And the same is true for series, that a series converges if and only if a series converges, now, starting at a different point along the sequence.

So this is the following theorem. Let xn be a sequence, and let capital M be a natural number. Then n equals 1 to infinity of x sub n.

This converges if and only if the sequence, now starting at capital M, converges, meaning when I have to decide whether a series converges or not, it doesn't matter what's going on for the first finitely many terms. What matters is what's going on as I keep adding terms from further and further out of this sequence.

And what's the proof? The proof is just expressing the partial sums for this guy in terms of the partial sums for this guy. So a partial sum satisfied for all m, sum from n equals 1 to m, and now x sub n as sum from n equals capital M to m x sub n plus sum from n equals 1 to capital M minus 1 x sub n.

So this is now just a fixed number. So this is a sequence of partial sums corresponding to this series. This is a sequence of partial sums corresponding to this series. And this is just a fixed number.

Therefore, if this converges, then this side converges to this plus this-- so maybe I'm going a little quick. But so if this converges, then this converges to this minus this number. And if this converges, then this sequence of partial sums converges to this limit plus this fixed number. And that's all I'm going to write.

Now, coming back to the usefulness of Cauchy sequences, this is kind of where they really become useful is in the study of series. Because again, it's difficult to sum. So when I keep talking saying the word sum a series, I'm talking about find the limit of partial sums. But because we have this equivalence between Cauchy sequences and convergent sequences, to decide if a series is convergent or not, we can just decide if, in some sense, it's Cauchy or not.

So let me make this definition. We say that a series x sub n is Cauchy if the sequence of partial sums-- again, I'm just going to put an m up top because this may start at 0 or n equals 1 or something-- so the sequence of partial sums is Cauchy.

And so let me just restate what we proved for sequences in terms of series. So we proved that every Cauchy sequence is convergent. So a series is Cauchy means that the sequence of partial sums is Cauchy. But we've proven that Cauchy sequences are convergent. So if this is Cauchy, then it's convergent and vice versa.

So based on what we've proven already for sequences, it follows that-- and this just follows immediately from what we've proven already, so I'm not even going to write a proof-- a series is Cauchy if and only if the series is convergent-- again, because both are defined in terms of the sequence of partial sums associated to the series. And we've already proven the equivalence between Cauchy and convergence for sequences.

Now, let me write what it means to be Cauchy in a slightly different way. And it's the following. So before, we had that a sequence is Cauchy, intuitively, if the elements of the sequence are getting close to each other. Now, for a series to be Cauchy, the intuitive way to think about it is that the tail of the sum is getting small, is getting arbitrarily small, the tail of the sum being if I add up finitely many numbers far enough out.

So a series is Cauchy if and only if all epsilon positive there exist to M, a natural number, such that for all integers l bigger than m, which is bigger than or equal to capital M, the sum from n equals n plus 1 to l of x sub n-- so this is a sum involving terms that are pretty far out there, at least all indexed by something bigger than or equal to capital M-- is less than epsilon. So a series is Cauchy if you're adding up smaller and smaller pieces, not individual pieces but actual adding those up.

So I'll leave it to you to do-- they're both pretty easy-- but this direction, I'll leave it to you as an exercise. And it'll follow immediately from what I'm going to write for, essentially, this direction. So let's suppose-- and let's make things concrete, starting somewhere-- let's suppose this sum is Cauchy. And we want to prove now that it has this property.

So let epsilon be positive. We now want to produce some capital number M so that this holds. So since the sequence of partial sums s sub m is Cauchy-- that's what it means for the series to be Cauchy-- there exists a natural number, capital M, such that for all M bigger than or equal to M0 and l bigger than or equal to M0, s sub m minus s sub l is less than epsilon.

So we're actually going to take capital M to be this M0. Choose M to be M0. Then, if l is bigger than m, which is bigger than or equal to M, which is equal to M0, if I look at this sum from n equals m plus 1 to l x sub n, I can write-- so this is absolute value. So in fact, let me remove the absolute value so that this becomes pretty clear.

I can write this sum as the l-th partial sum minus the n-th partial sum. Because the l-th partial sum sums from n equals 1 up to l. The m-th partial sums from n equals 1 up to m. So the sum containing only the terms between n plus 1 and l is the difference of these guys.

So now I'll put it on absolute values. And this thing, because l and m are bigger than or equal to capital M, which is equal to M0, and because I have this inequality, this is less than epsilon. And the converse direction, again, it follows immediately, essentially, from this equality here.

So to check whether a series converges, I don't have to somehow come up with a limit for this series, a sum for this series. I can just prove that the tail can be made arbitrarily small, as long as I go far enough out in the series.

And from this, we get a pretty simple elementary property, so a theorem. If a series converges, then this implies that the limit as n goes to infinity of x sub n of the sequence you used to obtain the series equals 0.

So this should fall in line with a series being convergent if and only if it satisfies this property here, which is somehow saying the tail of the sum is getting smaller and smaller, which means you can't be adding up big things as you go on out in the series.

So proof. So we'll show this by a simple epsilon M definition. So suppose xn converges. Then xn the series is Cauchy. And now I'm going to verify this using the epsilon M definition.

Let epsilon be positive. Since xn is Cauchy, this implies that there exists a natural number M0 such that for all l bigger than or equal to m bigger than or equal M0, I have that condition there, sum from n equals m plus 1 to l x sub n is less than epsilon.

Choose M to be M0 plus 1. So why M0 plus 1 but not exactly M0? Just because, basically, what I'm going to do is I'm going to take l to be equal to m plus 1. And the index gets shifted by 1.

Then if m is bigger than or equal to M, I get that the absolute value of x sub m-- maybe instead of saying limit as n goes to infinity, I'll write limit as m goes to infinity, it's just a change in the dummy variable-- this is equal to the sum from n equals m to m x sub n. And so I've shifted this.

So now little m is bigger than or equal to capital M0 plus 1. So you could write this as, if you like, m minus 1 plus 1. So little m minus 1 is bigger than or equal to M0. So by this inequality, this is less than epsilon.

So we see that if a series converges the terms, the individual terms, x sub n must converge to 0. So there's another reason why this series minus 1 to the n does not converge. Because those terms do not converge.

And this also tells us that for this geometric series, when r is greater than or equal to 1 in absolute value, the series does not converge. It does not converge. And the proof is-- and we proved this, in fact, I think, a few lectures ago, as well, that if the absolute value of r is bigger than 1, then the limit as n goes to infinity of r to the n, this limit does not exist.

We showed it's, in fact, unbounded. r to the n is an unbounded sequence. So it does not convert. So I'm using that theorem over there in a little bit roundabout way.

Let me restate this theorem over here. This theorem says if the series converges, then this limit equals 0. Now, this statement is logically equivalent to the negation of the converse, or there's an actual word for that, but I can't remember-- namely that the negation of this implies the negation of this.

So a logically equivalent way to rewrite that statement over there is that if this limit does not equal to 0-- so if it doesn't exist at all, that's also fine-- this implies that does not converge. So this restatement of the theorem over there is really what I used here.

All right. And I think I'll stop there. Next time, we'll see that this theorem here is a one-way street. And I think you covered this example in, probably, calculus, namely that one-way street in the sense that if x then converges, then this limit is 0. But the converse does not hold. Namely, it is not true that if this limit is 0, then this converges. And we'll see the famed harmonic series next time.