18.100A: Complete Lecture Notes

Lecture 1:
Sets, Set Operations, and Mathematical Induction

For this class, we will be using the book Introduction to Real Analysis, Volume I by Jiří Lebl **[L]**. I will use ■ to end proofs of examples, and □ to end proofs of theorems.

Basic Set Theory

Remark 1. *There are two main goals of this class:*

1. *Gain experience with proofs.*
2. *Prove statements about real numbers, functions, and limits.*

Sets

A set is a collection of objects called elements or members of that set. The empty set (denoted $\emptyset$) is the set with no elements. There are a few symbols that are super helpful to know as a shorthand, and will be used throughout the course. Let S be a set. Then

- $a \in S$ means that "a is an element in S."
- $a \notin S$ means that "a is not an element in S."
- $\forall$ means "for all."
- $:=$ means "define."
- $\exists$ means "there exists."
- $\exists!$ means "there exists a unique."
- $\implies$ means "implies."
- $\iff$ means "if and only if."

Definition 2 (Set Relations)
We want to relate different sets, and thus we get the following notation/definitions:

1. A set A is a subset of B, $A \subset B$, if every element of A is in B. Given $A \subset B$, if $a \in A \implies a \in B$.
2. Two sets A and B are equal, $A = B$, if $A \subset B$ and $B \subset A$.
3. A set A is a proper subset of B, $A \subsetneq B$ if $A \subset B$ and $A \neq B$.

One way we can describe a set is using "set building notation". We write

$$\{x \in A \mid P(x)\} \quad \text{or} \quad \{x \mid P(x)\}$$

to mean "all $x \in A$ that satisfies property $P(x)$". One example of this would be $\{x \mid x \text{ is an even number}\}$. There are a few key sets that we will use throughout this class:

1. The set of natural numbers: $\mathbb{N} = \{1, 2, 3, 4, \dots\}$.
2. The set of integers: $\mathbb{Z} = \{0, 1, -1, 2, -2, 3, -3, \dots\}$.

3. The set of rational numbers: $\mathbb{Q} = \{\frac{m}{n} \mid m, n \in \mathbb{Z} \text{ and } n \neq 0\}$.

4. The set of real numbers: $\mathbb{R}$.

It follows that

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}.$$

The fourth item on this list brings us to an important question, and the first goal of our course:

Problem 3

How do we describe $\mathbb{R}$?

We will answer this question in Lectures 3 and 4. In the meantime, let's continue our study of sets and proof methods. Given sets A and B, we have the following definitions:

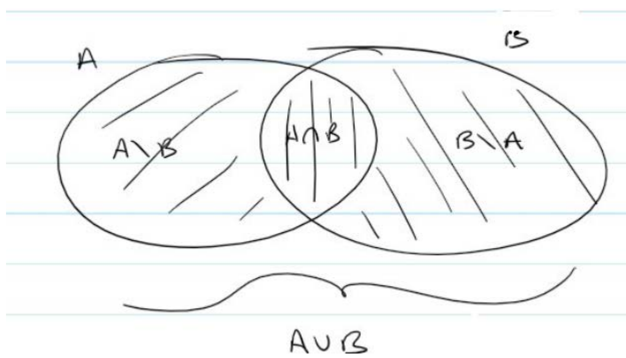
1. The union of A and B is the set $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$.

2. The intersection of A and B is the set $A \cap B = \{x \mid x \in A \text{ and } x \in B\}$.

3. The set difference of A and B is the set $A \setminus B = \{x \in A \mid x \notin B\}$.

4. The complement of A is the set $A^c = \{x \mid x \notin A\}$.

5. A and B are disjoint if $A \cap B = \emptyset$.



Theorem 4 (De Morgan's Laws)

If A, B, C are sets then

1. $(B \cup C)^c = B^c \cap C^c$,

2. $(B \cap C)^c = B^c \cup C^c$,

3. $A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$,

4. and $A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$.

We will prove the first statement to give an example of how such a proof would go, but the rest will be left to you.

Proof: Let B, C be sets. We must prove that

$$(B \cup C)^c \subset B^c \cap C^c \quad \text{and} \quad B^c \cap C^c \subset (B \cup C)^c.$$

If $x \in (B \cup C)^c \implies x \notin B \cup C \implies x \notin B$ and $x \notin C$. Hence, $x \in B^c$ and $x \in C^c \implies x \in B^c \cap C^c$. Thus, $(B \cup C)^c \subset B^c \cap C^c$.

If $x \in B^c \cap C^c$ then $x \in B^c$ and $x \in C^c \implies x \notin B$ and $x \notin C$. Hence, $x \notin B \cup C \implies x \in (B \cup C)^c$. Thus, $B^c \cap C^c \subset (B \cup C)^c$. □

Mathematical Induction

We will now talk about some of the biggest proof methods there are. Firstly, note that $\mathbb{N} = \{1, 2, 3, \dots\}$ has an ordering (as $1 < 2 < 3 < \dots$).

Axiom 5 (Well-ordering property)
The well-ordering property of $\mathbb{N}$ states that if $S \subset \mathbb{N}$ then there exists an $x \in S$ such that $x \leq y$ for all $y \in S$. In other words, there is always a smallest element.

Note that this is an axiom, and thus we have to assume this without proof.

Theorem 6 (Induction)
This concept was invented by Pascal in 1665. Let $P(n)$ be a statement depending on $n \in \mathbb{N}$. Assume that

1. (Base case) $P(1)$ is true and
2. (Inductive step) if $P(m)$ is true then $P(m+1)$ is true.

Then, $P(n)$ is true for all $n \in \mathbb{N}$.

Proof: Let $S = \{n \in \mathbb{N} \mid P(n) \text{ is not true}\}$. We wish to show that $S = \emptyset$. We will prove this by contradiction.

Remark 7. *When we prove something by contradiction, we assume the conclusion we want is false, and then show that we will reach a false statement. Rules of logic thus imply that the initial statement must be false. Thus in this case, we will assume $S \neq \emptyset$ and derive a false statement.*

Suppose that $S \neq \emptyset$. Then, by the well-ordering property of $\mathbb{N}$, S has a least element $m \in S$. Since $P(1)$ is true, $m \neq 1$, i.e. $m > 1$. Since m is a least element, $m - 1 \notin S \implies P(m-1)$ is true. This implies that $P(m)$ is true $\implies m \notin S$ by assumption. But then $m \in S$ and $m \notin S$. This is a contradiction. Thus $S = \emptyset$ and hence $P(n)$ is true for all $n \in \mathbb{N}$. □

Let's see an example of induction in action.

Theorem 8
For all $c \neq 1$ in the real numbers, and for all $n \in \mathbb{N}$,

$$1 + c + c^2 + \dots + c^n = \frac{1 - c^{n+1}}{1 - c}.$$

Proof: We will prove this by induction. First, we prove the base case ($n = 1$). The left hand side of the equation is $1 + c$ for $n = 1$. The right hand side is $\frac{1-c^2}{1-c} = \frac{(1-c)(1+c)}{1-c} = 1 + c$. Hence, the base case has been shown.

Assume that the equation is true for $k \in \mathbb{N}$, in other words

$$1 + c + c^2 + \dots + c^k = \frac{1 - c^{k+1}}{1 - c}.$$

Thus,

$$\begin{aligned}
\implies 1 + c + c^2 + \cdots + c^k + c^{k+1} &= (1 + c + c^2 + \cdots + c^k) + c^{k+1} \\
&= \frac{1 - c^{k+1}}{1 - c} + c^{k+1} \\
&= \frac{1 - c^{k+1} + c^{k+1}(1 - c)}{(1 - c)} \\
&= \frac{1 - c^{(k+1)+1}}{1 - c}.
\end{aligned}$$

Therefore, our proof is complete. □

Let's do another example:

Theorem 9

For all $c \geq -1$, $(1+c)^n \geq 1 + nc$ for all $n \in \mathbb{N}$.

Proof: We prove this through induction. In the base case, we have: $(1+c)^1 = 1 + 1 \cdot c$. For the inductive step, suppose that

$$(1+c)^m \geq 1 + mc.$$

Then,

$$(1+c)^{m+1} = (1+c)^m \cdot (1+c).$$

By assumption,

$$\begin{aligned}
&\geq (1 + mc) \cdot (1 + c) \\
&= 1 + (m+1)c + mc^2 \\
&\geq 1 + (m+1)c.
\end{aligned}$$

By induction, our proof is complete. □