18.100A: Complete Lecture Notes

Lecture 10:

The Completeness of the Real Numbers and Basic Properties of Infinite Series

Cauchy Sequences

Definition 1

Cauchy A sequence $\{x_n\}$ is Cauchy if $\forall \epsilon > 0\ \exists M \in \mathbb{N}$ such that for all $n, k \geq M$,

$$|x_n - x_k| < \epsilon.$$

Example 2

Show the sequence $x_n = \frac{1}{n}$ is Cauchy.

Proof: Let $\epsilon > 0$ and choose $M \in \mathbb{N}$ such that $\frac{1}{M} < \frac{\epsilon}{2}$. Then, if $n, k \geq M$, then

$$\left|\frac{1}{n} - \frac{1}{k}\right| \leq \frac{1}{n} + \frac{1}{k} \leq \frac{2}{M} < \epsilon.$$

■

Negation 3 (Not Cauchy)

By the negation of the definition, a sequence $\{x_n\}$ is not Cauchy if $\exists \epsilon_0 > 0$ such that for all $M \in \mathbb{N}$, $\exists n, k \geq M$ such that $|x_n - x_k| \geq \epsilon_0$.

Example 4

Show the sequence $x_n = (-1)^n$ is not Cauchy.

Proof: Choose $\epsilon = 1$ and let $M \in \mathbb{N}$. Choose $n = M$ and $k = M + 1$. Then,

$$|(-1)^n - (-1)^k| = 2 \geq 1.$$

■

Theorem 5

If $\{x_n\}$ is Cauchy, then $\{x_n\}$ is bounded.

Proof: If $\{x_n\}$ is Cauchy then $\exists M \in \mathbb{N}$ such that for all $n, k \geq M$,

$$|x_n - x_k| < 1.$$

Then, for all $n \geq M$, $|x_n - x_M| < 1$. Hence,

$$|x_n| \leq |x_n - x_M| + |x_M| < |x_M| + 1.$$

Let $B = |x_1| + \cdots + |x_M| + 1$. Then, for all $n \in \mathbb{N}$, $|x_n| \leq B$. □

Theorem 6

If $\{x_n\}$ is Cauchy and a subsequence $\{x_{n_k}\}$ converges, then $\{x_n\}$ converges.

Proof: Suppose that $\{x_{n_k}\}$ is a subsequence of $\{x_n\}$ such that $\lim_{k\to\infty} x_{n_k} = x$. We claim that $x_n \to x$. Let $\epsilon > 0$. Since $x_{n_k} \to x$, there exists $M_0 \in \mathbb{N}$ such that $\forall k \geq M_0$,

$$|x_{n_k} - x| < \frac{\epsilon}{2}.$$

Since $\{x_n\}$ is Cauchy, there exists an $M_1 \in \mathbb{N}$ such that for all $n \geq M_1$ and $m \geq M_1$,

$$|x_n - x_m| < \frac{\epsilon}{2}.$$

Choose $M = M_0 + M_1$. If $n \geq M$, then $n_M \geq M \geq M_0$ and $n_{\geq} M_1$. Therefore,

$$|x_n - x| \leq |x_n - x_{n_M}| + |x_{n_M} - x| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

□

Theorem 7

A sequence of real numbers $\{x_n\}$ is Cauchy if and only if $\{x_n\}$ is convergent.

Proof: ($\implies$) If $\{x_n\}$ is Cauchy, then $\{x_n\}$ is bounded. Therefore, $\{x_n\}$ has a convergent subsequence by Bolzano-Weierstrass. By the previous theorem, we thus have that $\{x_n\}$ is convergent.

($\impliedby$) Suppose that $\{x_n\}$ is convergent and $x = \lim_{n\to\infty} x_n$. Let $\epsilon > 0$. Since $x_n \to x$, $\exists M_0 \in \mathbb{N}$ such that $\forall n \geq M_0$,

$$|x_n - x| < \frac{\epsilon}{2}.$$

Choose $M = M_0$. Then, if $n, k \geq M$,

$$|x_n - x_k| \leq |x_n - x| + |x_k - x| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Therefore, $\{x_n\}$ is Cauchy. □

Series

Remark 8. *Series were the original motivation for analysis.*

Definition 9

Given $\{x_n\}$, the symbol $\sum_{n=1}^{\infty} x_n$ or $\sum x_n$ is the series associated to $\{x_n\}$. We say $\sum x_n$ converges if the sequence

$$\left\{ s_m = \sum_{n=1}^{m} x_n \right\}_{m=1}^{\infty}$$

converges. We call the terms of $\{s_m\}$ the partial sums. If $\lim_{m\to\infty} s_m = s$, we write $s = \sum x_n$ and treat $\sum x_n$ as a number.

Remark 10. *A series need not start at $n = 1$.*

Example 11

$\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ converges.

Proof: We may do show this directly by consider the partial sums:

$$\begin{aligned} s_m = \sum_{n=1}^{m} \frac{1}{n(n+1)} &= \sum_{n=1}^{m} \frac{1}{n} - \frac{1}{n+1} \\ &= \left(1 + \frac{1}{2} + \cdots + \frac{1}{m}\right) - \left(\frac{1}{2} + \ldots \frac{1}{m} + \frac{1}{m+1}\right) \\ &= 1 - \frac{1}{m+1}. \end{aligned}$$

Thus, $s_m = 1 - \frac{1}{m+1} \to 1$. Hence, the partial sums converge and thus the series converges. ■

Theorem 12

If $|r| < 1$ then $\sum_{n=0}^{\infty} r^n$ converges and

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}.$$

Proof: We have $\forall m \in \mathbb{N}$,

$$s_m = \sum_{n=0}^{m} r^n = \frac{1 - r^{m+1}}{1-r}$$

by induction. Since $|r| < 1$, $\lim_{m\to\infty} |r|^{m+1} = 0$. Therefore,

$$\lim_{m\to\infty} s_m = \frac{1-0}{1-r} = \frac{1}{1-r}.$$

□

Remark 13. *Series of the form $\sum_{n=0}^{\infty} \alpha(r)^n$ for $\alpha \in \mathbb{R}$ and $r \in \mathbb{R}$ are called geometric series.*

Theorem 14

Let $\{x_n\}$ be a sequence and let $M \in \mathbb{N}$. Then, $\sum_{n=1}^{\infty} x_n$ converges if and only if $\sum_{n=M}^{\infty} x_n$ converges.

Proof: The partial sums satisfy, for all $m \in \mathbb{N}$,

$$\sum_{n=1}^{m} x_n = \sum_{n=M}^{m} x_n + \sum_{n=1}^{M} x_n.$$

□

Definition 15

$\sum x_n$ is Cauchy if the sequence of partial sums is Cauchy.

Theorem 16

$\sum x_n$ is Cauchy $\iff$ $\sum x_n$ is convergent.

Proof: This follows by the analogous theorem for regular sequences of real numbers proven earlier. □

Theorem 17

$\sum x_n$ is Cauchy if and only if $\forall \epsilon > 0$, $\exists M \in \mathbb{N}$ such that for all $m \geq M$ and $\ell > m$,

$$\left| \sum_{n=m+1}^{\ell} x_n \right| < \epsilon.$$

Proof: ($\implies$) Suppose $\sum x_n$ is Cauchy. Let $\epsilon > 0$. Then, $\exists M_0 \in \mathbb{N}$ such that $\forall m, \ell \geq M_0$,

$$|s_m - s_\ell| < \epsilon.$$

Choose $M = M_0$. Then, if $m \geq M$ and $\ell > m$, then

$$\left| \sum_{n=m+1}^{\ell} x_n \right| = |s_\ell - s_m| < \epsilon.$$

The other direction is left as an exercise. □

Theorem 18

If $\sum x_n$ converges then $\lim_{n\to\infty} x_n = 0$.

Proof: Suppose $\sum x_n$ converges. Then, $\sum x_n$ is Cauchy. Let $\epsilon > 0$. Since $\sum x_n$ is Cauchy, $\exists M_0 \in \mathbb{N}$ such taht for all $\ell > m \geq M_0$,

$$\left| \sum_{n=m+1}^{\ell} x_n \right| < \epsilon.$$

Choose $M = M_0 + 1$. Then, if $m \geq M \implies m - 1 \geq M_0$. Therefore,

$$|x_m| = \left| \sum_{n=m}^{m} x_n \right| < \epsilon$$

by taking $\ell = m$. □

Theorem 19

If $|r| \geq 1$, then $\sum_{n=0}^{\infty} r^n$ diverges.

Proof: If $|r| \geq 1$, then $\lim_{m\to\infty} r^m \neq 0$. Therefore, $\sum_{n=0}^{\infty} r^n$ diverges, as if this wasn't the case then $\lim_{m\to\infty} r^m = 0$ by the previous theorem which is a contradiction. □

Corollary 20

The series $\sum_{n=0}^{\infty} \alpha(r)^n$ converges if and only if $|r| < 1$.