18.100A: Complete Lecture Notes

Lecture 16:
The Min/Max Theorem and Bolzano's Intermediate Value Theorem

As we will see in today's lecture, continuous functions are well behaved on closed intervals of the form $[a, b]$, with $f([a, b]) = [e, f]$ for some $e, f \in \mathbb{R}$.

Definition 1 (Bounded Functions)
A function $f : S \to \mathbb{R}$ is bounded if $\exists B \geq 0$ such that for all $x \in S$,

$$|f(x)| \leq B.$$

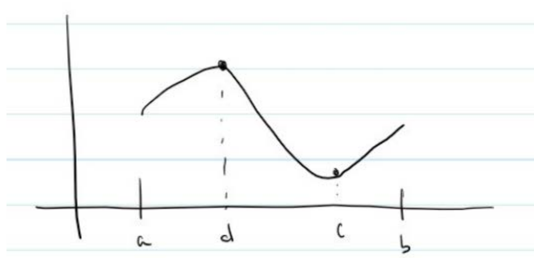
Theorem 2
If $f : [a, b] \to \mathbb{R}$ is continuous then f is bounded.

Proof: Suppose for the sake of contradiction that $f : [a, b] \to \mathbb{R}$ is continuous and f is unbounded. Then, $\forall n \in \mathbb{N}$, $\exists x_n \in [a, b]$ such that $|f(x_n)| \geq n$. By the Bolzano-Weierstrass theorem, $\exists$ a subsequence $\{x_{n_k}\}_k$ of $\{x_n\}_n$ and an $x \in \mathbb{R}$ such that $x_{n_k} \to x$. Since $a \leq x_{n_k} \leq b$ for all k, $a \leq x \leq b$. Given f is continuous at x by assumption,

$$f(x) = \lim_{k\to\infty} f(x_{n_k}) \implies |f(x)| = \lim_{k\to\infty} |f(x_{n_k})|.$$

Therefore, $\{|f(x_{n_k})|\}$ is bounded, and thus $\{n_k\}$ is bounded since $n_k \leq |f(x_{n_k})|$. But by the definition of a subsequence, we must have $k \leq n_k$ for all k, contradicting the boundedness of $\{n_k\}$. $\square$

Definition 3 (Absolute Minimum/Maximum)
Let $f : S \to \mathbb{R}$. Then, f achieves an absolute minimum at c if $\forall x \in S$, $f(x) \geq f(c)$. Similarly, f achieves an absolute maximum at d if $\forall x \in S$, $f(x) \leq f(d)$.



Theorem 4 (Min-Max Theorem)
Let $f : [a, b] \to \mathbb{R}$. If f is continuous, then f achieves an absolute maximum and absolute minimum.

Remark 5. *Note that this is also called the Extreme Value Theorem or EVT for short, though to stay consistent with the Lebl's book I will be calling it the Min-Max theorem.*

Proof: We will prove this for the absolute maximum. If f is continuous, then f is bounded by the previous theorem. Thus, the set

$$E = \{f(x) \mid x \in [a, b]\}$$

is bounded above. Let $L = \sup E$. Then,

1. L is an upper bound for E, i.e.

$$\forall x \in [a, b],\ \ f(x) \leq L.$$

2. There exists a sequence $\{f(x_n)\}_n$ with $x_n \in [a, b]$ such that $f(x_n) \to L$.

By the Bolzano-Weierstrass theorem, there exists a subsequence $\{x_{n_k}\}_k$ of $\{x_n\}$ and $d \in [a, b]$ such that $x_{n_k} \to d$ as $k \to \infty$. Hence,

$$f(d) = \lim_{k\to\infty} f(x_{n_k}) = \lim_{n\to\infty} f(x_n) = L$$

by the continuity of f. Thus, f achieves an absolute maximum at d.

We leave the absolute minimum proof to the reader. □

Remark 6. *As students of mathematics, we also care about the necessity of the hypotheses!*

For example, what if $f : [a, b] \to \mathbb{R}$ is not continuous? Does the Min-Max theorem apply? The answer is **no**. Consider

$$f(x) = \begin{cases} \frac{1}{2} & x = 0, 1 \\ x & x \in (0, 1) \end{cases}.$$

Here, f neither achieves an absolute maximum nor an absolute minimum on $[0, 1]$.

What if $f : S \to \mathbb{R}$ and S is not closed and bounded? Does the Min-Max theorem apply? Again, the answer is **no**. Consider $f(x) = \frac{1}{x} - \frac{1}{1-x}$ on $S = (0, 1)$. Even though f is continuous on S, f neither achieves an absolute minimum nor an absolute maximum.

So far we have shown that if $f : [a, b] \to \mathbb{R}$ is continuous, then $f([a, b]) \subset [f(c), f(d)]$ where f achieves an absolute minimum at c and an absolute maximum at d.

Question 7. *Does f achieve all values in $f(c)$ and $f(d)$?*

The answer is **yes**, by Bolzano's Intermediate Value Theorem as we will show.

Theorem 8

Let $f : [a, b] \to \mathbb{R}$. If $f(a) < 0$ and $f(b) > 0$, then $\exists c \in (a, b)$ such that $f(c) = 0$.

Proof: We prove this using a bisection method. Let $a_1 = a$ and $b_1 = b$, and define a_2, b_2 as follows: If $f((a_1 + b_1)/2) \geq 0$, define $a_2 = a_1$, $b_2 = \frac{a_1+b_1}{2}$. If $f((a_1 + b_1)/2) < 0$, define $a_2 = \frac{a_1+b_1}{2}$ and $b_2 = b_1$. In general, if we know a_n, b_n, we choose a_{n+1} and b_{n+1} as follows: If $f((a_n + b_n)/2) \geq 0$, define $a_{n+1} = a_n$, $b_2 n + 1 = \frac{a_n+b_n}{2}$. If $f((a_n + b_n)/2) < 0$, define $a_{n+1} = \frac{a_n+b_n}{2}$ and $b_{n+1} = b_n$. Thus, we have:

1. $\forall n \in \mathbb{N}$, $a \leq a_n \leq a_{n+1} \leq b_{n+1} \leq b_n \leq b$.
2. $\forall n \in \mathbb{N}$, $b_{n+1} - a_{n+1} = \frac{b_n - a_n}{2}$.
3. $\forall n \in \mathbb{N}$, $f(a_n) < 0$ and $f(b_n) \geq 0$.

By 1., $\{a_n\}$ and $\{b_n\}$ are monotone increasing and monotone decreasing respectively, both of which are bounded. Thus, $\exists c, d \in [a,b]$ such that $a_n \to c$ and $b_n \to d$. By 2.,

$$b_n - a_n = \frac{b_{n-1} - a_{n-1}}{2} = \frac{1}{4}(b_{n-2} - a_{n-2}) = \cdots = \frac{1}{2^{n-1}}(b-a).$$

Thus,

$$d - c = \lim_{n\to\infty}(b_n - a_n) = \lim_{n\to\infty}\frac{1}{2^{n-1}}(b-a) = 0 \implies d = c.$$

Therefore, $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = c$. By 3., $f(c) = \lim_{n\to\infty} f(a_n) \leq 0$ and $f(c) = \lim_{n\to\infty} f(b_n) \geq 0$. Therefore, $f(c) = 0$. □

Theorem 9 (Bolzano IVT)
Let $f : [a,b] \to \mathbb{R}$ be continuous. If $f(a) < f(b)$, and $y \in (f(a), f(b))$, $\exists c \in (a,b)$ such that $f(c) = y$. If $f(b) < f(a)$ and $y \in (f(b), f(a))$, $\exists c \in (a,b)$ such that $f(c) = y$.

Remark 10. *This is known as the Intermediate Value Theorem or IVT for short.*

Proof: Suppose $f(a) < f(b)$. Let $y \in (f(a), f(b))$. Define $g(x) = f(x) - y$. Then, $g : [a,b] \to \mathbb{R}$ is continuous, $g(a) = f(a) - y < 0$ and $g(b) = f(b) - y > 0$. Therefore, by the previous theorem, $\exists c \in (a,b)$ such that $g(c) = 0$. Therefore, $\exists c \in (a,b)$ such that $g(c) = f(c) - y = 0 \implies f(c) = y$.

The other direction is analogous. □

Theorem 11
Let $f : [a,b] \to \mathbb{R}$ be continuous. let $c \in [a,b]$ be where f achieves an absolute minimum and $d \in [a,b]$ be where f achieves an absolute maximum. Then,

$$f([a,b]) = [f(c), f(d)].$$

In other words, every value between the absolute minimum value and the absolute maximum value is achieved.

Proof: We know that $f([a,b]) \subseteq [f(c), f(d)]$. Hence, we prove the other direction. By the IVT applied to $f : [c,d] \to \mathbb{R}$,

$$[f(c), f(d)] \subseteq f([c,d]) \subseteq f([a,b]).$$

Therefore, $f([a,b]) = [f(c), f(d)]$. □

Of course, Bolzano IVT is false if we assume f is not continuous (as can be seen by the following diagram):

Theorem 12
The polynomial $f(x) = x^{2021} + x^{2020} + 9.03x + 1$ has at least one real root.

Proof: Notice that $f(0) = 1 > 0$ and $f(-1) = -1 + 1 - 9.03 + 1 = -8.03 < 0$. Thus, by IVT, $\exists c \in (-1, 0)$ such that $f(c) = 0$. $\square$