# 18.100A: Complete Lecture Notes 

## Lecture 16:

## The Min/Max Theorem and Bolzano's Intermediate Value Theorem

As we will see in today's lecture, continuous functions are well behaved on closed intervals of the form $[a, b]$, with $f([a, b])=[e, f]$ for some $e, f \in \mathbb{R}$.

Definition 1 (Bounded Functions)
A function $f: S \rightarrow \mathbb{R}$ is bounded if $\exists B \geq 0$ such that for all $x \in S$,

$$
|f(x)| \leq B
$$

## Theorem 2

If $f:[a, b] \rightarrow \mathbb{R}$ is continuous then $f$ is bounded.

Proof: Suppose for the sake of contradiction that $f:[a, b] \rightarrow \mathbb{R}$ is continuous and $f$ is unbounded. Then, $\forall n \in \mathbb{N}$, $\exists x_{n} \in[a, b]$ such that $\left|f\left(x_{n}\right)\right| \geq n$. By the Bolzano-Weierstrass theorem, $\exists$ a subsequence $\left\{x_{n_{k}}\right\}_{k}$ of $\left\{x_{n}\right\}_{n}$ and an $x \in \mathbb{R}$ such that $x_{n_{k}} \rightarrow x$. Since $a \leq x_{n_{k}} \leq b$ for all $k, a \leq x \leq b$. Given $f$ is continuous at $x$ by assumption,

$$
f(x)=\lim _{k \rightarrow \infty} f\left(x_{n_{k}}\right) \Longrightarrow|f(x)|=\lim _{k \rightarrow \infty}\left|f\left(x_{n_{k}}\right)\right| .
$$

Therefore, $\left\{\left|f\left(x_{n_{k}}\right)\right|\right\}$ is bounded, and thus $\left\{n_{k}\right\}$ is bounded since $n_{k} \leq\left|f\left(x_{n_{k}}\right)\right|$. But by the definition of a subsequence, we must have $k \leq n_{k}$ for all $k$, contradicting the boundedness of $\left\{n_{k}\right\}$.

## Definition 3 (Absolute Minimum/Maximum)

Let $f: S \rightarrow \mathbb{R}$. Then, $f$ achieves an absolute minimum at $c$ if $\forall x \in S, f(x) \geq f(c)$. Similarly, $f$ achieves an absolute maximum at $d$ if $\forall x \in S, f(x) \leq f(d)$.


## Theorem 4 (Min-Max Theorem)

Let $f:[a, b] \rightarrow \mathbb{R}$. If $f$ is continuous, then $f$ achieves an absolute maximum and absolute minimum.

Remark 5. Note that this is also called the Extreme Value Theorem or EVT for short, though to stay consistent with the Lebl's book I will be calling it the Min-Max theorem.

Proof: We will prove this for the absolute maximum. If $f$ is continuous, then $f$ is bounded by the previous theorem. Thus, the set

$$
E=\{f(x) \mid x \in[a, b]\}
$$

is bounded above. Let $L=\sup E$. Then,

1. $L$ is an upper bound for $E$, i.e.

$$
\forall x \in[a, b], f(x) \leq L
$$

2. There exists a sequence $\left\{f\left(x_{n}\right)\right\}_{n}$ with $x_{n} \in[a, b]$ such that $f\left(x_{n}\right) \rightarrow L$.

By the Bolzano-Weierstrass theorem, there exists a subsequence $\left\{x_{n_{k}}\right\}_{k}$ of $\left\{x_{n}\right\}$ and $d \in[a, b]$ such that $x_{n_{k}} \rightarrow d$ as $k \rightarrow \infty$. Hence,

$$
f(d)=\lim _{k \rightarrow \infty} f\left(x_{n_{k}}\right)=\lim _{n \rightarrow \infty} f\left(x_{n}\right)=L
$$

by the continuity of $f$. Thus, $f$ achieves an absolute maximum at $d$.
We leave the absolute minimum proof to the reader.
Remark 6. As students of mathematics, we also care about the necessity of the hypotheses!
For example, what if $f:[a, b] \rightarrow \mathbb{R}$ is not continuous? Does the Min-Max theorem apply? The answer is no. Consider

$$
f(x)= \begin{cases}\frac{1}{2} & x=0,1 \\ x & x \in(0,1)\end{cases}
$$

Here, $f$ neither achieves an absolute maximum nor an absolute minimum on $[0,1]$.
What if $f: S \rightarrow \mathbb{R}$ and $S$ is not closed and bounded? Does the Min-Max theorem apply? Again, the answer is no. Consider $f(x)=\frac{1}{x}-\frac{1}{1-x}$ on $S=(0,1)$. Even though $f$ is continuous on $S, f$ neither achieves an absolute minimum nor an absolute maximum.

So far we have shown that if $f:[a, b] \rightarrow \mathbb{R}$ is continuous, then $f([a, b]) \subset[f(c), f(d)]$ where $f$ achieves an absolute minimum at $c$ and an absolute maximum at $d$.

Question 7. Does $f$ achieve all values in $f(c)$ and $f(d)$ ?
The answer is yes, by Bolzano's Intermediate Value Theorem as we will show.

## Theorem 8

Let $f:[a, b] \rightarrow \mathbb{R}$. If $f(a)<0$ and $f(b)>0$, then $\exists c \in(a, b)$ such that $f(c)=0$.

Proof: We prove this using a bisection method. Let $a_{1}=a$ and $b_{1}=b$, and define $a_{2}, b_{2}$ as follows: If $f\left(\left(a_{1}+\right.\right.$ $\left.\left.b_{1}\right) / 2\right) \geq 0$, define $a_{2}=a_{1}, b_{2}=\frac{a_{1}+b_{1}}{2}$. If $f\left(\left(a_{1}+b_{1}\right) / 2\right)<0$, define $a_{2}=\frac{a_{1}+b_{1}}{2}$ and $b_{2}=b_{1}$. In general, if we know $a_{n}, b_{n}$, we choose $a_{n+1}$ and $b_{n+1}$ as follows: If $f\left(\left(a_{n}+b_{n}\right) / 2\right) \geq 0$, define $a_{n+1}=a_{n}, b_{2} n+1=\frac{a_{n}+b_{n}}{2}$. If $f\left(\left(a_{n}+b_{n}\right) / 2\right)<0$, define $a_{n+1}=\frac{a_{n}+b_{n}}{2}$ and $b_{n+1}=b_{n}$. Thus, we have:

1. $\forall n \in \mathbb{N}, a \leq a_{n} \leq a_{n+1} \leq b_{n+1} \leq b_{n} \leq b$.
2. $\forall n \in \mathbb{N}, b_{n+1}-a_{n+1}=\frac{b_{n}-a_{n}}{2}$.
3. $\forall n \in \mathbb{N}, f\left(a_{n}\right)<0$ and $f\left(b_{n}\right) \geq 0$.

By 1., $\left\{a_{n}\right\}$ and $\left\{b_{n}\right\}$ are monotone increasing and monotone decreasing respectively, both of which are bounded. Thus, $\exists c, d \in[a, b]$ such that $a_{n} \rightarrow c$ and $b_{n} \rightarrow d$. By 2 .,

$$
b_{n}-a_{n}=\frac{b_{n-1}-a_{n-1}}{2}=\frac{1}{4}\left(b_{n-2}-a_{n-2}\right)=\cdots=\frac{1}{2^{n-1}}(b-a)
$$

Thus,

$$
d-c=\lim _{n \rightarrow \infty}\left(b_{n}-a_{n}\right)=\lim _{n \rightarrow \infty} \frac{1}{2^{n-1}}(b-a)=0 \Longrightarrow d=c
$$

Therefore, $\lim _{n \rightarrow \infty} a_{n}=\lim _{n \rightarrow \infty} b_{n}=c$. By 3., $f(c)=\lim _{n \rightarrow \infty} f\left(a_{n}\right) \leq 0$ and $f(c)=\lim _{n \rightarrow \infty} f\left(b_{n}\right) \geq 0$. Therefore, $f(c)=0$.

## Theorem 9 (Bolzano IVT)

Let $f:[a, b] \rightarrow \mathbb{R}$ be continuous. If $f(a)<f(b)$, and $y \in(f(a), f(b)), \exists c \in(a, b)$ such that $f(c)=y$. If $f(b)<f(a)$ and $y \in(f(b), f(a)), \exists c \in(a, b)$ such that $f(c)=y$.

Remark 10. This is known as the Intermediate Value Theorem or IVT for short.
Proof: Suppose $f(a)<f(b)$. Let $y \in(f(a), f(b))$. Define $g(x)=f(x)-y$. Then, $g:[a, b] \rightarrow \mathbb{R}$ is continuous, $g(a)=f(a)-y<0$ and $g(b)=f(b)-y>0$. Therefore, by the previous theorem, $\exists c \in(a, b)$ such that $g(c)=0$. Therefore, $\exists c \in(a, b)$ such that $g(c)=f(c)-y=0 \Longrightarrow f(c)=y$.

The other direction is analogous.

## Theorem 11

Let $f:[a, b] \rightarrow \mathbb{R}$ be continuous. let $c \in[a, b]$ be where $f$ achieves an absolute minimum and $d \in[a, b]$ be where $f$ achieves an absolute maximum. Then,

$$
f([a, b])=[f(c), f(d)] .
$$

In other words, every value between the absolute minimum value and the absolute maximum value is achieved.

Proof: We know that $f([a, b]) \subseteq[f(c), f(d)]$. Hence, we prove the other direction. By the IVT applied to $f:[c, d] \rightarrow \mathbb{R}$,

$$
[f(c), f(d)] \subseteq f([c, d]) \subseteq f([a, b])
$$

Therefore, $f([a, b])=[f(c), f(d)]$.
Of course, Bolzano IVT is false if we assume $f$ is not continuous (as can be seen by the following diagram):


## Theorem 12

The polynomial $f(x)=x^{2021}+x^{2020}+9.03 x+1$ has at least one real root.

Proof: Notice that $f(0)=1>0$ and $f(-1)=-1+1-9.03+1=-8.03<0$. Thus, by IVT, $\exists c \in(-1,0)$ such that $f(c)=0$.

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