18.100A: Complete Lecture Notes

Lecture 19:
Differentiation Rules, Rolle's Theorem, and the Mean Value Theorem

Theorem 1

Let $f : I \to \mathbb{R}$, $g : I \to \mathbb{R}$ be differentiable at $c \in I$. Then,

1. (Linearity) $\forall \alpha \in \mathbb{R}$, $(\alpha f + g)'(c) = \alpha f'(c) + g'(c)$.
2. (Product rule) $(fg)'(c) = f'(c)g(c) + f(c)g'(c)$.
3. (Quotient rule) If $g(x) \neq 0$ for all $x \in I$, then
$$\left(\frac{f}{g}\right)'(c) = \frac{f'(c)g(c) - f(c)g'(c)}{(g(c))^2}.$$

Proof:

1. We can compute this directly:
$$\lim_{x\to c} \frac{(\alpha f + g)(x) - (\alpha f + g)(c)}{x - c} = \lim_{x\to c} \alpha \frac{f(x) - f(c)}{x - c} + \frac{g(x) - g(c)}{x - c} = \alpha f'(x) + g'(x).$$
2. We first write
$$\frac{f(x)g(x) - f(c)g(c)}{x - c} = \frac{f(x) - f(c)}{x - c} \cdot g(x) + f(c) \cdot \frac{g(x) - g(c)}{x - c}$$
and use the fact that $\lim_{x\to c} g(x) = f(c)$.
3. The quotient rule is left as an exercise to the reader.

$\square$

Theorem 2 (Chain Rule)

Let I_1, I_2 be two intervals, $g : I_1 \to I_2$ be differentiable at $c \in I_1$, and $f : I_2 \to \mathbb{R}$ differentiable at $g(c)$. Then, $f \circ g : I_1 \to \mathbb{R}$ is differentiable at c and
$$(f \circ g)'(c) = f'(g(c))g'(c).$$

Proof: Let $h(x) = f(g(x))$ and $d = g(c)$. We want to prove that $h'(c) = f'(d)g'(c)$. Define the following
$$u(y) = \begin{cases} \frac{f(y)-f(d)}{y-x} & y \neq d \\ f'(d) & y = d \end{cases} \quad \text{and} \quad v(y) = \begin{cases} \frac{g(x)-g(c)}{x-c} & x \neq c \\ g'(d) & x = c \end{cases}.$$

Then,
$$\lim_{y\to d} u(y) = \lim_{y\to d} \frac{f(y) - f(d)}{y - d} = f'(d) = u(d).$$

Similarly,

$$\lim_{x \to c} v(x) = \lim_{x \to c} \frac{g(x) - g(c)}{x - c} = g'(c) = v(c).$$

In other words, u is continuous at d and v is continuous at c. Now,

$$\begin{aligned} f(y) - f(d) &= u(y)(y - d) \\ g(x) - g(c) &= v(x)(x - c) \\ \implies h(x) - h(c) &= f(g(x)) - f(d) \\ &= u(g(x))(g(x) - g(c)) \\ &= u(g(x))v(x)(x - c). \end{aligned}$$

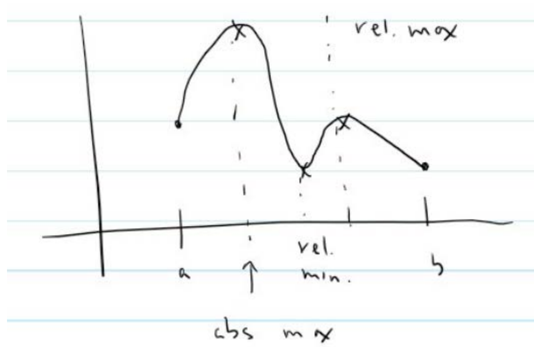
Therefore,

$$\begin{aligned} \lim_{x \to c} \frac{h(x) - h(c)}{x - c} &= \lim_{x \to c} u(g(x))v(x) \\ &= u(g(c))v(c) \\ &= f'(g(c))g'(c). \end{aligned}$$

□

Mean Value Theorem

Definition 3 (Relative Maximum/Minimum)
Let $S \subset \mathbb{R}$ and $f : S \to \mathbb{R}$. Then, f has a relative maximum at $c \in S$ if $\exists \delta > 0$ such that for all $x \in S$, $|x - c| < \delta \implies f(x) \leq f(c)$. The definition for relative maximum is analogous.



Theorem 4
If $f : [a, b] \to \mathbb{R}$ has a relative max or min at $c \in (a, b)$ and f is differentiable at c, then

$$f'(c) = 0.$$

Proof: If f has a relative maximum at $c \in (a, b)$ then $\exists \delta > 0$ such that $(c - \delta, c + \delta) \subset (a, b)$ and $\forall x \in (c - \delta, c + \delta)$, $f(x) \leq f(c)$. Let

$$x_n = c - \frac{\delta}{2n} \in (c - \delta, c).$$

Then, $x_n \to c$ so

$$f'(c) = \lim_{n \to \infty} \frac{f(x_n) - f(c)}{x_n - c} \geq 0.$$

Now let

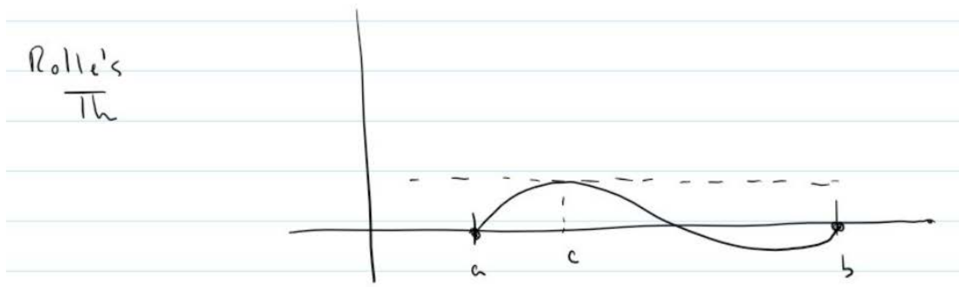
$$y_n = c + \frac{\delta}{2n} \in (c, c+\delta).$$

Then, $y_n \to c$ so

$$f'(c) = \lim_{n\to\infty} \frac{f(y_n) - f(c)}{y_n - c} \le 0.$$

Therefore, $f'(c) = 0$. The proof for relative minimum is similar and thus left to the reader. $\square$

Theorem 5 (Rolle)
Let $f : [a, b] \to \mathbb{R}$ be continuous and differentiable on (a, b). If $f(a) = f(b)$, then $\exists c \in (a, b)$ such that $f'(c) = 0$.



Remark 6. *Are the hypotheses all necessary? This is left to the reader to figure out.*

Proof: Let $K = f(a) = f(b)$. Since f is continuous on $[a, b]$, $\exists c_1, c_2 \in [a, b]$ such that f achieves an absolute maximum at c_1 and absolute minimum at c_2. If $f(c_1) > K \implies c_1 \in (a, b)$. Therefore, $f'(c_1) = 0$ by the previous theorem. Similarly, if $f(c_2) < K$, then $c_2 \in (a, b) \implies f'(c_2) = 0$. If

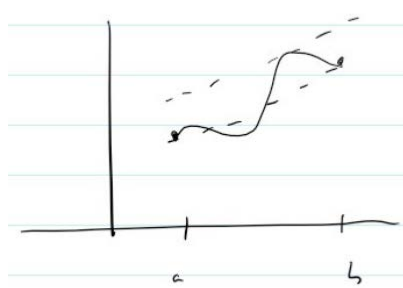
$$f(c_1) \le K \le f(c_2) \implies f(x) = K \ \forall x \in [a, b] \implies f'(c) - 0 \ \text{ for any } \ c \in (a, b).$$

$\square$

Theorem 7 (Mean Value Theorem)
Let $f : [a, b] \to \mathbb{R}$ be continuous, and let f be differentiable on (a, b). Then, $\exists c \in (a, b)$ such that

$$f(b) - f(a) = f'(c)(b - a).$$

Remark 8. *The Mean Value Theorem is sometimes denoted MVT.*



Proof: Define $g : [a, b] \to \mathbb{R}$ with

$$g(x) = f(x) - f(b) + \frac{f(b) - f(a)}{b - a}(b - x).$$

Then, $g(a) = g(b) = 0$. Thus, by Rolle's theorem, $\exists c \in (a, b)$ with $g'(c) = 0$, and hence

$$0 = f'(c) - \frac{f(b) - f(a)}{b - a}.$$

$\square$

We now look at some useful applications of the MVT.

Theorem 9

If $f : I \to \mathbb{R}$ is differentiable and $f'(x) = 0$ for all $x \in I$, then f is constant.

Proof: Let $a, b \in I$ with $a < b$. Then, f is continuous on $[a, b]$ and differentiable on (a, b). Therefore, $\exists c \in (a, b)$ such that $f(b) - f(a) = (b - a)f'(c) = 0$. Hence, $f(b) = f(a)$ for all $a, b \in I$ such that $a < b$. $\square$

Theorem 10

Let $f : I \to \mathbb{R}$ be differentiable. Then,

1. f is increasing if and only if $f'(x) \geq 0$ for all $x \in I$ and
2. f is decreasing if and only if $f'(x) \leq 0$ for all $x \in I$.

Proof:

1. ($\Longleftarrow$) Suppose $f'(x) \geq 0$ for all $x \in I$. Let $a, b \in I$ with $a < b$. Then, by MVT, $\exists c \in (a, b)$ such that

$$f(b) - f(a) = (b - a)f'(c) \geq 0 \implies f(a) \leq f(b).$$

   ($\Longrightarrow$) Suppose f is increasing. Let $c \in I$ and let $\{x_n\}$ be a sequence in I such that $x_n \to c$ such that $\forall n$, $x_n < c$. Then, for all n, $f(x_n) - f(c) \leq 0 \implies \frac{f(x_n)-f(c)}{x_n-c} \geq 0$. Therefore,

$$f'(c) = \lim_{n\to\infty} \frac{f(x_n) - f(c)}{x_n - c} \geq 0.$$

   Now let $\{x_n\}$ be a sequence in I such that $x_n \to c$ such that $\forall n$, $x_n > c$. Then, for all n, $f(x_n) - f(c) \geq 0 \implies \frac{f(x_n)-f(c)}{x_n-c} \geq 0$. Therefore,

$$f'(c) = \lim_{n\to\infty} \frac{f(x_n) - f(c)}{x_n - c} \geq 0.$$

   Hence, in either case, $f'(c) \geq 0$.

2. Notice that f is decreasing if and only if $-f$ is increasing, and apply 1. to $-f$.

$\square$