18.100A: Complete Lecture Notes

Lecture 22:
The Fundamental Theorem of Calculus, Integration by Parts, and Change of Variable Formula

Theorem 1 (Additivity)
If $f \in C([a, b])$ and $a < c < b$, then
$$\int_a^b f = \int_a^c f + \int_c^b f.$$

Proof: Let $\{(\underline{y}(r), \underline{\zeta}(r))\}_r$ and $\{(\underline{z}(r), \underline{\eta}(r))\}_r$ be tagged partitions of $[a, c]$ and $[c, b]$ respectively such that $\|\underline{y}(r)\| \to 0$ and $\|\underline{z}(r)\| \to 0$. Define

$$\underline{x}(r) = \underline{y}(r) \cup \underline{z}(r)$$
$$\underline{\xi}(r) = \underline{\zeta}(r) \cup \underline{\eta}(r),$$

a sequence of tagged partitions of $[a, b]$. Then,

$$\|\underline{x}(r)\| \leq \|\underline{y}(r)\| + \|\underline{z}(r)\| \to 0.$$

Thus,

$$\begin{aligned} \int_a^b f &= \lim_{t\to\infty} S_f(\underline{x}(r), \underline{\xi}(r)) \\ &= \lim_{r\to\infty} (S_f(\underline{y}(r), \underline{\zeta}(r)) + S_f(\underline{z}(r), \underline{\eta}(r))) \\ &= \int_a^c f + \int_c^b f. \end{aligned}$$

□

Theorem 2
Let $f \in C([a, b])$, and

$$m_f = \inf\{f(x) \mid x \in [a, b]\} \in \mathbb{R}$$
$$M_f = \sup\{f(x) \mid x \in [a, b]\} \in \mathbb{R}.$$

Then,
$$m_f(b - a) \leq \int_a^b f \leq M_f(b - a).$$

Proof: Let $\{(\underline{x}(r), \underline{\xi}(r))\}_r$ be a sequence of tagged partitions with $\|\underline{x}(r)\| \to 0$. Then,

$$S_f(\underline{x}(r), \underline{\xi}(r)) = \sum_{k=1}^n f(\xi_k(r))(x_k(r) - x_{k-1}(r)) \geq m_f \sum_{k=1}^n (x_k(r) - x_{k-1}(r)) = m_f(b - a).$$

Similarly,

$$S_f(\underline{x}(r), \underline{\xi}(r)) = \sum_{k=1}^{n} f(\xi_k(r))(x_k(r) - x_{k-1}(r)) \leq M_f \sum_{k=1}^{n}(x_k(r) - x_{k-1}(r)) = M_f(b-a).$$

Therefore, for all r,

$$m_f(b-a) \leq S_f(\underline{x}(r), \underline{\xi}(r)) \leq M_f(b-a) \implies m_f(b-a) \leq \int_a^b f \leq M_f(b-a).$$

□

Theorem 3

Suppose $f \in C([a,b])$ and $g \in C([a,b])$.

1. If $\forall x \in [a,b]\ f(x) \leq g(x)$, then
$$\int_a^b f \leq \int_a^b g.$$
2. (Triangle Inequality for integrals): $|\int_a^b f| \leq \int_a^b |f|$.

Proof:

1. Let $\{(\underline{x}(r), \underline{\xi}(r)\}_r$ be a sequence of tagged partitions such that $\|\underline{x}(r)\| \to 0$. Then, for all $r \in \mathbb{N}$,
$$\begin{aligned} S_f(\underline{x}(r), \underline{\xi}(r)) &= \sum_{j=1}^{n} f(\xi_j(r))(x_j(r) - x_{j-1}(r)) \\ &\leq \sum_{j=1}^{n} g(\xi_j(r))(x_j(r) - x_{j-1}(r)) \\ &= S_g(\underline{x}(r), \underline{\xi}(r)). \end{aligned}$$
Then, letting $r \to \infty$, we get that
$$\int_a^b f \leq \int_a^b g.$$
2. Notice that $\pm f(x) \leq |f(x)|$ for all x, and thus
$$\pm \int_a^b f \leq \int_a^b |f| \implies -\int_a^b f \leq \int_a^b f \leq \int_a^b |f|.$$
Therefore, $|\int_a^b f| \leq \int_a^b |f|$.

□

Remark 4. *There are some conventions that are worth noting:*

1. $\int_a^a f := 0$. *This is consistent with our definitions and theorems thus far as* $\lim_{b \to a} |\int_a^b f| = 0$.
2. $\int_a^b f = -\int_b^a f$.

Fundamental Theorem of Calculus

Theorem 5 (Fundamental Theorem of Calculus)
Suppose $f \in C([a,b])$.

1. If $F : [a,b] \to \mathbb{R}$ is differentiable and $F' = f$, then
$$\int_a^b f = F(b) - F(a).$$
2. The function $G(x) := \int_a^x f$ is differentiable on $[a,b]$ and
$$\begin{cases} G' = f \\ G(a) = 0 \end{cases}.$$

Remark 6. *We sometimes abbreviate the Fundamental Theorem of Calculus to FTC.*

Proof:

1. Let $\{(\underline{x}(r))\}_r$ be a sequence of partitions with $\|\underline{x}\| \to 0$. Then, by the Mean Value Theorem, $\forall r \forall j$, there exists a $\xi_j(r) \in [x_{j-1}(r), x_j(r)]$ such that
$$F(x_j(r)) - F(x_{j-1}(r)) = F'(\xi_j(r))(x_j(r) - x_{j-1}(r)) = f(\xi_j(r))(x_j(r) - x_{j-1}(r)).$$
Thus,
$$\begin{aligned} \int_a^b f &= \lim_{r\to\infty} \sum_{j=1}^{n(r)} f(\xi_j(r))(x_j(r) - x_{j-1}(r)) \\ &= \lim_{r\to\infty} \sum_{j=1}^{n(r)} F(x_j(r)) - F(x_{j-1}(r)) \\ &= \lim_{r\to\infty} (F(b) - F(a)) = F(b) - F(a). \end{aligned}$$
2. Let $c \in [a,b]$. We wish to show that
$$\lim_{x\to c} \frac{\int_a^x f - \int_a^c f}{x - c} = f(c).$$
Let $\epsilon > 0$. Then, since f is continuous at c, $\exists \delta_0 > 0$ such that
$$|t - c| < \delta_0 \implies |f(t) - f(c)| < \epsilon/2.$$
Choose $\delta = \delta_0$. Suppose $0 < x - c < \delta$. If $t \in [c,x]$, then
$$|t - c| \leq |x - c| < \delta = \delta_0.$$

Thus,

$$\begin{aligned}
\left|\frac{1}{x-c}\int_c^x f(t)\,\mathrm{d}t - f(c)\right| &= \left|\frac{1}{x-c}\int_c^t f(t)\,\mathrm{d}t - \frac{1}{x-c}\int_x^c f(c)\,\mathrm{d}t\right| \\
&= \frac{1}{x-c}\left|\int_c^x (f(t)-f(c))\,\mathrm{d}t\right| \\
&\leq \frac{1}{x-c}\int_c^x |f(t)-f(c)|\,\mathrm{d}t \\
&\leq \frac{1}{x-c}\int_c^x \epsilon/2\,\mathrm{d}t \\
&= \frac{1}{x-c}\cdot\frac{\epsilon}{2}(x-c) = \frac{\epsilon}{2}.
\end{aligned}$$

A similar argument holds for $0 < c - x < \delta$. Thus,

$$0 < |x-c| < \delta \implies \left|\frac{\int_a^x f - \int_a^c f}{x-c} - f(c)\right| \leq \frac{\epsilon}{2} < \epsilon.$$

□

Theorem 7 (Integration by Parts)
Suppose $f, g \in C([a,b])$ and $f', g' \in C([a,b])$. Then,

$$\int_a^b f'g = (f(b)g(b) - f(a)g(a)) - int_a^b fg'.$$

Proof: We have

$$(fg)' = f'g + fg'.$$

Therefore, by the Fundamental Theorem of Calculus,

$$f(b)g(b) - f(a)g(a) = \int_a^b f'g + \int_a^b fg'.$$

□

Remark 8. *We sometimes abbreviate Integration By Parts as IBP.*

Lemma 9 (Riemann-Lebesgue)
Suppose $f \in C([-\pi,\pi])$ and $f' \in C([-\pi,\pi])$ with f 2π-periodic with $f(-\pi) = f(\pi)$. For $n \in \mathbb{N} \cup \{0\}$, let

$$\begin{aligned}
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,\mathrm{d}x \\
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,\mathrm{d}x.
\end{aligned}$$

Then,

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = 0.$$

Definition 10 (Fourier Coefficients)
The a_n, b_n defined in the above lemma are referred to as the Fourier coefficients of f.

Proof: Using IBP, we have

$$\begin{aligned}
|b_n| &= \frac{1}{\pi}\left|\int_{-\pi}^{\pi} f(x)\cos(nx)\,\mathrm{d}x\right| \\
&= \frac{1}{\pi}\left|\int_{-\pi}^{\pi} (\frac{1}{n}\sin(nx))' f(x)\,\mathrm{d}x\right| \\
&= \left|\frac{1}{n}(f(\pi)\sin(n\pi) - f(-pi)\sin(n(-\pi))) - \frac{1}{n}\int_{-\pi}^{\pi}\sin(nx)f'(x)\,\mathrm{d}x\right|.
\end{aligned}$$

Notice that $\sin(n\pi) = \sin(n(-\pi)) = 0$ for all $n \in \mathbb{N}$. Hence,

$$\begin{aligned}
|b_n| &\leq \frac{1}{n}\int_{-\pi}^{\pi} |\sin(nx)||f'(x)|\,\mathrm{d}x \\
&\leq \frac{1}{n}\int_{-\pi}^{\pi} |f'| \to 0.
\end{aligned}$$

By the Squeeze Theorem, $|b_n| \to 0$. A similar arguments works for a_n. ☐

Theorem 11 (Change of Variables)

Let $\varphi : [a,b] \to [c,d]$ be continuously differentiable with $\varphi' > 0$ on $[a,b]$, $\varphi(a) = c$, and $\varphi(b) = d$. Then,

$$\int_c^d f(u)\,\mathrm{d}u = \int_a^b f(\varphi(x))\varphi'(x)\,\mathrm{d}x.$$

Proof: Let $F : [a,b] \to \mathbb{R}$ such that $F' = f$. Then,

$$F(\varphi(x))' = f(\varphi(x)).$$

Hence, by the FTC,

$$\begin{aligned}
\int_a^b f(\varphi(x))\varphi'(x)\,\mathrm{d}x &= \int_a^b F(\varphi(x))'\,\mathrm{d}x \\
&= F(\varphi(b)) - F(\varphi(a)) \\
&= F(d) - F(c).
\end{aligned}$$

Furthermore, by the FTC,

$$\int_c^d f(u)\,\mathrm{d}u = \int_c^d F(u)'\,\mathrm{d}u = F(d) - F(c).$$

☐