Sequences of Function

Power Series

Remark 1. Power series motivate the general discussion of sequences of functions.

Definition 2 (Power series)
A power series about $x_0$ is a series of the form
\[ \sum_{m=0}^{\infty} a_m(x-x_0)^m. \]

Theorem 3
Suppose
\[ R = \lim_{m \to \infty} |a_m|^{1/n} \]
exists, and let
\[ p = \begin{cases} \frac{1}{R} & R > 0 \\ \infty & R = 0. \end{cases} \]
Then, $\sum a_m(x-x_0)^m$ converges absolutely if $|x-x_0| < p$ and diverges if $|x-x_0| > p$.

Definition 4 (Radius of Convergence)
In the above theorem, we define $p$ to be the radius of convergence.

Proof: We have
\[ \lim_{n \to \infty} |a_m(x-x_0)^m|^{1/m} = R|x-x_0|, \]
and the theorem follows by the Root test.

Suppose $\sum a_m(x-x_0)^m$ is a power series with radius of convergence $p$. Furthermore, define $f : (x_0 - p, x_0 + p) \to \mathbb{R}$ such that
\[ f(x) := \sum_{m=0}^{\infty} a_m(x-x_0)^m. \]
Then, $f$ is a limit of a sequence of functions
\[ f(x) = \lim_{n \to \infty} f_n(x), \]
for \( x \in (x_0 - p, x_0 + p) \) and where

\[
f_n(x) = \sum_{m=0}^{n} a_m(x - x_0)^m.
\]

**Example 5**

For example, we have

\[
f(x) = \frac{1}{1-x} = \sum_{m=0}^{\infty} x^m.
\]

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**Question 6.** This concept begs a number of questions:

1. Is \( f \) continuous?

2. Is \( f \) differentiable, and does \( f' = \lim_{n \to \infty} f'_n \)?

3. If 1. is true, does

\[
\int_a^b f = \lim_{n \to \infty} \int_a^b f_n?
\]

These questions will be the key motivator for the last section of this course.

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**Pointwise and Uniform Convergence**

We now consider a setting far more general than power series.

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**Definition 7** (Pointwise Convergence)

For \( n \in \mathbb{N} \), let \( f_n : S \to \mathbb{R} \). Let \( f : S \to \mathbb{R} \). We say that \( \{f_n\} \) converges pointwise to \( f \) if for all \( x \in S \),

\[
\lim_{n \to \infty} f_n(x) = f(x).
\]

Let’s consider some examples.

1. Let \( f_n(x) = x^n \) on \([0,1]\). Then,

\[
\lim_{n \to \infty} f_n(x) = \begin{cases} 0 & x \in [0,1) \\ 1 & x = 1 \end{cases}.
\]

Thus, \( \{f_n\} \) converges to the above pointwise function. Hence, notice that a sequence of continuous functions may not converge pointwise to a continuous function!

2. Let \( f_n(x) = \sum_{m=0}^{n} x^m \) for \( x \in (-1,1) \). Then,

\[
\lim_{n \to \infty} f_n(x) = \lim_{n \to \infty} \sum_{m=0}^{n} x^m = \frac{1}{1 - x}.
\]

Hence, pointwise, this sequence converges to its power series (see the above example).

3. Let \( f_n(x) : [0,1] \to \mathbb{R} \) be defined by

\[
f_n(x) = \begin{cases} 4n^2x & x \in [0, \frac{1}{2n}] \\ 4n - 4n^2x & x \in \left[ \frac{1}{2n}, \frac{1}{n} \right] \\ 0 & x \in \left[ \frac{1}{n}, 1 \right] \end{cases}.
\]

We can picture this sequence (on the next page)
Then, \( \lim_{n \to \infty} f_n(0) = \lim_{n \to \infty} 0 = 0 \). Let \( x \in (0, 1] \). Let \( N \in \mathbb{N} \) such that \( \frac{1}{N} < x \). Then, for all \( n \geq N \),

\[
f_n(x) = 0.
\]

Therefore,

\[
\{f_n(x)\} = f_1(x), \ldots, f_{N-1}(x), 0, 0, 0, \ldots
\]

Hence, \( \lim_{n \to \infty} f_n(x) = 0 \) for all \( x \in [0, 1] \). Thus, \( \{f_n\} \) converges pointwise to \( f(x) = 0 \) on \([0, 1]\).

**Definition 8 (Uniform Convergence)**

For \( n \in \mathbb{N} \), let \( f_n : S \to \mathbb{R} \), and let \( f : S \to \mathbb{R} \). Then, we say \( f_n \) converges to \( f \) uniformly or **converges uniformly to** \( f \) if \( \forall \epsilon > 0 \ \exists M \in \mathbb{N} \) such that for all \( n \geq M \ \forall x \in S \),

\[
|f_n(x) - f(x)| < \epsilon
\]

**Theorem 9**

If \( f_n : S \to \mathbb{R}, f : S \to \mathbb{R} \), and \( f_n \to f \) uniformly, then \( f_n \to f \) pointwise.

**Proof:** Let \( c \in S \) and let \( \epsilon > 0 \). Then, \( f_n \to f \) uniformly implies that there exists \( M_0 \in \mathbb{N} \) such that for all \( n \geq M, \forall x \in S, |f_n(x) - f(x)| < \epsilon \). Choose \( M = M_0 \). Then, \( \forall n \geq M \),

\[
|f_n(c) - f(c)| < \epsilon.
\]

Thus, \( \lim_{n \to \infty} f_n(c) = f(c) \) for all \( c \in S \), and therefore \( f_n \to f \) pointwise. \( \square \)