18.100A: Complete Lecture Notes

Lecture 24:
Uniform Convergence, the Weierstrass M-Test, and Interchanging Limits

Theorem 1

Let $f_n(x) = x^n$, and let $f(x) = \begin{cases} 0 & x \in [0,1) \\ 1 & x = 1 \end{cases}$.

1. $\forall 0 < b < 1$, $f_n \to f$ uniformly on $[0,b]$.
2. f_n does not converge to f uniformly on $[0,1]$.

Proof:

1. Let $\epsilon > 0$. Since $b \in (0,1)$, $b^n \to 0$. Therefore, $\exists M_0 \in \mathbb{N}$ such that for all $n \geq M_0$, $b^n < \epsilon$. Choose $M = M_0$. Then, $\forall n \geq M, \forall x \in [0,b]$,
$$|f_n(x) - f(x)| = |f_n(x)| = x^n \leq b^n < \epsilon.$$
Thus, $f_n \to f$ uniformly on $[0,b]$.

Before proving the other part, we first note the following negation:

Negation 2 (Not Uniformly Convergent)

$f_n : S \to \mathbb{R}$ does not converge to $f : S \to \mathbb{R}$ uniformly if $\exists \epsilon_0 > 0$ such that $\forall M \in \mathbb{N}$, $\exists n \geq M$ and $\exists x \in S$ with $|f_n(x) - f(x)| \geq \epsilon_0$.

2. Hence, for our example, choose $\epsilon_0 = \frac{1}{4}$. Let $M \in \mathbb{N}$ and choose $x = \left(\frac{1}{2}\right)^{\frac{1}{M}} \in (0,1)$. Thus,
$$|f_M(x) - f(x)| = f_M(x) = \frac{1}{2} > \epsilon_0.$$

□

Theorem 3 (Weierstrass M-test)

let $f_j : S \to \mathbb{R}$ and suppose $\exists M_j > 0$ such that

a) $\forall x \in S$, $|f_j(x)| \leq M_j$.

b) $\sum_{j=1}^{\infty} M_j$ converges.

Then,

1. $\forall x \in S$, $\sum_{j=1}^{\infty} f_j(x)$ converges absolutely.
2. Let $f(x) = \sum_{j=1}^{\infty} f_j(x)$ for $x \in S$. Then,
$$\sum_{j=1}^{n} f_j \to f \quad \text{uniformly on} \quad S.$$

Proof:

1. The first part follows from a), b), and the Comparison Test.
2. Let $\epsilon > 0$. Since $\sum M_j$ converges, $\exists N_0 \in \mathbb{N}$ such that $\forall n \geq N_0$,

$$\sum_{j=n+1}^{\infty} M_j = \left| \sum_{j=1}^{\infty} M_j - \sum_{j=1}^{n} M_j \right| < \epsilon.$$

Choose $N = N_0$. Then, for all $n \geq N$ and $\forall x \in S$,

$$\begin{aligned} \left| f(x) - \sum_{j=1}^{n} f_j(x) \right| &= \left| \sum_{j=n+1}^{\infty} f_j(x) \right| \\ &\leq \sum_{j=n+1}^{\infty} |f_j(x)| \\ &\leq \sum_{j=n+1}^{\infty} M_j < \epsilon. \end{aligned}$$

Thus, $\sum_{j=1}^{n} f_j \to f$ uniformly on S.

□

Interchange of Limits

Remark 4. *In general, limits cannot be interchanged.*

Example 5

For instance, consider the following example:

$$\begin{aligned} \lim_{n\to\infty} \lim_{k\to\infty} \frac{n/k}{n/k+1} &= \lim_{n\to\infty} \frac{0}{0+1} = 0 \\ \lim_{k\to\infty} \lim_{n\to\infty} \frac{n/k}{n/k+1} &= \lim_{k\to\infty} 1 = 1. \end{aligned}$$

Question 6. *Hence, we ask three questions about interchanging limits:*

1. *If $f_n : S \to \mathbb{R}$, f_n continuous and $f_n \to f$ pointwise or uniform, then is f continuous?*
2. *If $f_n : [a, b] \to \mathbb{R}$, f_n differentiable, and $f_n \to f$ with $f_n' \to g$, then is f differentiable and does $g = f'$?*
3. *If $f_n : [a, b] \to \mathbb{R}$, with f_n and f continuous such that $f_n \to f$, then does*

$$\int_a^b f_n = \int_a^b f?$$

The answer to the above questions are all **yes**, if the convergence is uniform.

Question 7. *What if the convergence is only pointwise?*

If convergence is only pointwise, the answer to the above questions are all **no**, which we will show next time.

18.100A / 18.1001 Real Analysis
Fall 2020