18.100A: Complete Lecture Notes

Lecture 5:

The Archimedian Property, Density of the Rationals, and Absolute Value

For all $x, y \in \mathbb{R}$ and x < y, there exists an $r \in \mathbb{R}$ such that x < r < y (take $r = \frac{x+y}{2}$).

Question 1. Can we find $r \in \mathbb{Q}$ such that x < r < y?

Theorem 2

The answer is yes!

- i) (Archimedian Property) If $x, y \in \mathbb{R}$ and x > 0, then $\exists n \in \mathbb{N}$ such that nx > y.
- ii) (Density of \mathbb{Q}) If $x, y \in \mathbb{R}$ and x < y then $\exists r \in \mathbb{Q}$ such that x < r < y.

Proof:

i) Suppose that $x, y \in \mathbb{R}$ and x > 0. Then we wish to show that $\exists n \in \mathbb{N}$ such that $n > \frac{y}{x}$. Suppose this is not the case. Then, $\forall n \in \mathbb{N}, n \leq \frac{y}{x}$. In other words, \mathbb{N} is bounded above by $\frac{y}{x}$. Hence, $\exists a = \sup \mathbb{N} \in \mathbb{R}$. Since a is the least upper bound for $\mathbb{N}, a - 1$ cannot be an upper bound for \mathbb{N} . Hence, $\exists m \in \mathbb{N}$ such that

$$a - 1 < m \implies a < m + 1 \in \mathbb{N}.$$

However, this is a contradiction, because then a is not an upper bound for N. Therefore, $\exists n \in \mathbb{N}$ such that $n \geq \frac{y}{x}$.

- ii) Suppose $x, y \in \mathbb{R}$ and x < y. Then, there are three cases:
 - $0 \le x < y$,
 - x < 0 < y, and
 - $x < y \le 0$.

For the second case, take $r = 0 \in \mathbb{Q}$. So, assume that $0 \le x < y$. Then, by the Archimedian Property, $\exists n \in \mathbb{N}$ such that n(y-x) > 1. Again by the Archimedean property, $\exists l \in \mathbb{N}$ such that l > nx. Thus, consider the set

$$S = \{k \in \mathbb{N} \mid k > nx\}.$$

By the well-ordering property of \mathbb{N} , S has a least element, $m \in S \implies nx < m \implies x < \frac{m}{n} \in \mathbb{Q}$. Since $m - 1 \notin S$, $m - 1 \le nx \implies m \le nx + 1 < ny$. Hence, $\frac{m}{n} < y$. Therefore,

$$x < \frac{m}{n} < y$$

If instead we have $x < y \le 0$, then $0 \le -y < -x \implies \exists \tilde{r} \in \mathbb{Q}$ such that

$$-y < \tilde{r} < x \implies x < -\tilde{r} < y$$

by the previous case.

Theorem 3 $1 = \sup \left\{ 1 - \frac{1}{n} \mid n \in \mathbb{N} \right\}.$

Proof: If $n \in \mathbb{N}$, then $1 - \frac{1}{n} < 1 \implies 1$ is an upper bound of this set. Suppose that x is an upper bound for the set $\{1 - 1/n \mid n \in \mathbb{N}\}$. We now prove that $x \ge 1$. For the sake of contradiction, assume that x < 1. By the Archimedean property, there exists an $n \in \mathbb{N}$ such that 1 < n(1 - x). Therefore, $\exists n \in \mathbb{N}$ such that x < 1 - 1/n. Hence, x is not an upper bound for the set $\{1 - 1/n \mid n \in \mathbb{N}\}$ if x < 1. Thus, if x is an upper bound, $x \ge 1$. Therefore,

$$\sup\left\{1-\frac{1}{n}\mid n\in\mathbb{N}\right\}=1.$$

We now begin proving some theorems about supremums and infinimums which will make them easier to use.

Theorem 4

Suppose that $S \subset \mathbb{R}$ is nonempty and bounded above. Then, $x = \sup S$ if and only if

- 1. x is an upper bound for S.
- 2. for all $\epsilon > 0$, $\exists y \in S$ such that $x \epsilon < y \le x$.

Proof: This is left as an exercise in Assignment 3.

Notation 5

For $x \in \mathbb{R}$ and $A \subset \mathbb{R}$, define

$$x + A := \{x + a \mid a \in A\}$$
$$xA := \{xa \mid a \in A\}.$$

Theorem 6

Using this new notation, we have the following theorems:

1. If $x \in \mathbb{R}$ and A is bounded above, then x + A is bounded above and

$$\sup(x+A) = x + \sup A.$$

2. If x > 0 and A is bounded above then xA is bounded above and

 $\sup(xA) = x \sup A.$

Proof:

1. Suppose that $x \in \mathbb{R}$ and A is bounded above. Therefore, $\sup A \in \mathbb{R}$ by the least upper bound property of \mathbb{R} . Then, $\forall a \in A, a \leq \sup A$. Hence,

$$\forall a \in A, \ x + a \le x + \sup A.$$

Hence, $x + \sup A$ is an upper bound for x + A. Let $\epsilon > 0$. Then, $\exists y \in A$ such that

 $\sup A - \epsilon < y \le \sup A \implies (x + \sup A) - \epsilon < y + x \le x + \sup A.$

Therefore, by our previous theorem, $x + \sup A = \sup(x + A)$.

2. Suppose that x > 0 and A is bounded above. Thus, $\sup A \in \mathbb{R}$. Then, $\forall a \in A, a \leq \sup A$ and thus $xa \leq x \sup A$. Hence, $x \sup A$ is an upper bound of xA. Let $\epsilon > 0$. Then $\exists y \in A$ such that

$$\sup A - \frac{\epsilon}{x} < y \le \sup A \implies x \sup A - \epsilon < xy \le x \sup A.$$

Therefore, by the previous theorem, $\sup(xA) = x \sup A$.

Theorem 7

Let $A, B \subset \mathbb{R}$ such that $\forall x \in A, \forall y \in B, x \leq y$. Then, $\sup A \leq \inf B$.

Proof: The proof of this is left to the reader. <u>Absolute Value</u>

Definition 8

If $x \in \mathbb{R}$ we define

$$|x| := \begin{cases} x, & x \ge 0\\ -x, & x \le 0 \end{cases}.$$

Theorem 9

We can prove a bunch of theorems about the absolute value function that we usually take for granted:

|x| ≥ 0 and |x| = 0 ⇔ x = 0.
∀x ∈ ℝ, |-x| = |x|.
∀x, y ∈ ℝ, |xy| = |x||y|.
|x²| = x² = |x|².
If x, y ∈ ℝ, then |x| ≤ y ⇔ -y ≤ x ≤ y.
∀x ∈ ℝ, x ≤ |x|.

Proof:

1) If $x \ge 0$ then $|x| = x \ge 0$. If $x \le 0$, then $-x \ge 0 \implies |x| = -x \ge 0$. Thus, $|x| \ge 0$. Now suppose x = 0. Then, |x| = x = 0. For the other direction, suppose |x| = 0. Then, if $x \ge 0 \implies x = |x| = 0$. If $x \le 0$, then -x = |x| = 0. Therefore, $x = 0 \iff |x| = 0$.

2) If $x \ge 0$ then $-x \le 0$. Thus, |x| = x = -(-x) = |-x|. If $x \le 0$ then $-x \ge 0$ and thus |-x| = |-(-x)| = |x|.

3) If $x \ge 0$ and $y \ge 0$, then $xy \ge 0$ and |xy| = xy = |x||y|. If $x \le 0$ and $y \le 0$, then

$$xy \le 0 \implies |xy| = -xy = (-x)y = |x||y|.$$

- 4) Take x = y in 3). Then, $|x^2| = |x|^2$. Since $x^2 \ge 0$, it follows that $|x^2| = x^2$.
- 5) Suppose $|x| \le y$. If $x \ge 0$, then $-y \le 0 \le x = |x| \le y$. Therefore, $-y \le x \le y$. If $x \le 0$, then $-x \ge 0$ and $|-x| \le y$. Hence, $-y \le -x \le y \implies -y \le x \le y$.

6) Take y = |x| in 5).

prove the *extremely* important Triangle Inequality.

On its own, these properties of the absolute values may not seem all that useful, but in the next lecture we will

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