18.100A: Complete Lecture Notes

Lecture 6:
The Uncountabality of the Real Numbers

Theorem 1 (Triangle Inequality)
$\forall x, y \in \mathbb{R}$,

$$|x+y| \leq |x| + |y|.$$

Proof: Let $x, y \in \mathbb{R}$. Then, $x + y \leq |x| + |y|$ and

$$(-x) + (-y) \leq |-x| + |-y| = |x| + |y|.$$

Therefore, $-(|x|+|y|) \leq x + y \leq |x| + |y|$. Hence,

$$|x+y| \leq |x| + |y|$$

by our previous theorem. □

Remark 2. *We may denote the Triangle Inequality with $\triangle$-inequality as a shorthand.*

Question 3. *As we showed in Assignment 1, we know that $\mathbb{Q}$ is countable. Is the set of real numbers countable?*

Recall 4
Recall that a set A is countable if A is either finite or $|A| = |\mathbb{N}|$.

We can think of $\mathbb{Q}$ as decimal expansions. In other words, we can think of a rational number x as being in the form

$$x = 10^k d_k + \cdots + 10 d_1 + d_0 + 10^{-1} d_{-1} + \cdots + 10^{-M} d_{-M}$$

with $d_i \in \{0, 1, 2, 3, \ldots, 9\}$. We may write

$$x = d_k d_{k-1} \ldots d_1 d_0 \bullet d_{-1} \ldots d_{-M}$$

where $\bullet$ is the decimal point. The same can be said about real numbers if we allow for infinite decimal expansions.

Definition 5
Let $x \in (0, 1]$ and let $d_{-j} \in \{0, 1, \ldots, 9\}$. We say that x is **represented** by the digits $\{d_{-j} \mid j \in \mathbb{N}\}$, i.e. $x = 0 \bullet d_{-1} d_{-2} \ldots$, if

$$x = \sup\{10^{-1} d_{-1} + 10^{-2} d_{-2} + \cdots + 10^{-n} d_{-n} \mid n \in \mathbb{N}\}.$$

Here is an example: $.2500 = \sup\{2 \cdot 10^{-1}, 2 \cdot 10^{-1} + 5 \cdot 10^{-2}, 2 \cdot 10^{-1} + 5 \cdot 10^{-2} + 0 \cdot 10^{-3}, \ldots\}$. Notice here that after a while the previous set becomes $\frac{1}{4}$ repeating. Hence, we have $.2500 = \sup\left\{\frac{1}{5}, \frac{1}{4}\right\} = \frac{1}{4}$.

Theorem 6
For every $x \in (0,1]$, there exists a unique sequence of digits d_{-j} such that $x = 0 \bullet d_{-1}d_{-2}\dots$ and

$$0 \bullet d_{-1}d_{-2}\dots d_{-n} < x \leq 0 \bullet d_{-1}d_{-2}\dots d_{-n} + 10^{-n}.$$

Furthermore, for every set of digits $\{d_{-j} \mid j \in \mathbb{N}\}$, there exists a unique $x \in [0,1]$ such that $x = 0 \bullet d_{-1}\dots$.

Notice however that the representative of $\frac{1}{2}$ is $0.4999\dots$.

Theorem 7 (Cantor)
(0,1] is uncountable.

Proof: We will prove this through contradiction. Suppose that $(0,1]$ is countable. Therefore, there exists a bijection $x : \mathbb{N} \to (0,1]$. We now construct a $y \in (0,1]$ such that y is not in the range of x. We write

$$x(n) = 0 \bullet d_{-1}^{(n)}d_{-2}^{(n)}\dots.$$

These are not exponents! This is the set of digits for a given $n \in \mathbb{N}$. In other words, x takes in a natural number n and maps it to the sequence of digits $\left\{d_{-j}^{(n)} \mid n \in \mathbb{N}\right\}$. Let

$$e_{-j} = \begin{cases} 1, & d_{-j}^{(j)} \neq 1 \\ 2, & d_{-j}^{(j)} = 1 \end{cases}.$$

Let $y = 0 \bullet e_{-1}e_{-2}\dots$. Then, $\forall n \in \mathbb{N}$,

$$0 \bullet e_{-1}e_{-2}\dots e_{-n} \leq y 0 \bullet e_{-1}\dots e_{-n} + 10^{-n}$$

since all e_{-j}s are positive. Thus, $0 \bullet e_{-1}\dots$ is the unique decimal expansion of y. However, for all $n \in \mathbb{N}$, $d_{-n}^{(n)} \neq e_{-n}$. Therefore, $\forall n$, $x(n) \neq y$. This is a contradiction, and thus $(0,1]$ is uncountable. □

So $(0,1] \subset \mathbb{R}$ is uncountable!

Corollary 8
The set of real numbers, $\mathbb{R}$, is uncountable.

Sequences and Series

Remark 9. *Analysis is the study of limits.*

Sequences and Limits

Definition 10 (Sequence of Reals)
A sequence of real numbers is a function $x : \mathbb{N} \to \mathbb{R}$. We denote $x(n) = x_n$ and we denote the sequence by $\{x_n\}_{n=1}^{\infty}$, $\{x_n\}$, or $x_1, x_2, \dots$.

Definition 11
A sequence $\{x_n\}$ is bounded if $\exists B \geq 0$ such that $\forall n$, $|x_n| \leq B$.

One example of a bounded sequence is $x_n = \frac{1}{n}$, since $\left|\frac{1}{n}\right| \leq 1$ for all $n \in \mathbb{N}$. However, $x_n = n$ is not bounded.

Remark 12. *A sequence is different from a set!*

For example,

$$-1, 1, -1, 1, \cdots = \{(-1)^n\}_{n=1}^{\infty},$$

while

$$\{(-1)^n \mid n \in \mathbb{N}\} = \{-1, 1\}.$$

Definition 13 (Sequence Convergence of Reals)
A sequence $\{x_n\}$ converges to $x \in \mathbb{R}$ if $\forall \epsilon > 0$, $\exists M \in \mathbb{N}$ such that $\forall n \geq M$,

$$|x_n - x| < \epsilon.$$

A sequence that converges is said to be **convergent**, and otherwise is said to be **divergent**. We can also define divergence as the negation of convergent.

Negation 14 (Not Convergent)
The sequence $\{x_n\}$ is not convergent, or divergent if $\exists \epsilon_0 > 0$ such that $\forall M \in \mathbb{N}$, $\exists n \geq M$ so that

$$|x_n - x| \geq \epsilon_0.$$

We now prove two theorems:

Theorem 15
If $\{x_n\}$ converges for x and y, then $x = y$. In other words, limits of convergent sequences of real numbers are unique.

Theorem 16
Let $x, y \in \mathbb{R}$. If $\forall \epsilon > 0$, $|x - y| < \epsilon$, then $x = y$.

Proof: We first prove the second theorem. Suppose that $x \neq y$. Then, $|x - y| > 0$. Hence, choosing $\epsilon = \frac{|x-y|}{2}$, we have

$$|x - y| \leq \frac{|x - y|}{2} \implies \frac{|x - y|}{2} < 0$$

which is a contradiction. □

Using this we prove the former theorem. Suppose x_n converges to x and to y. We will show that for all $\epsilon > 0$, $|x - y| < \epsilon$. Firstly, given $x_n \to x$, for $\epsilon > 0$ there exists an $N_1 \in \mathbb{N}$ such that $\forall n \geq N_1$,

$$|x_n - x| < \frac{\epsilon}{2}.$$

Then, given $x_n \to y$, for $\epsilon > 0$ there exists an $N_2 \in \mathbb{N}$ such that $\forall n \geq N_2$,

$$|x_n - y| < \frac{\epsilon}{2}.$$

Let $N = \max\{N_1, N_2\}$. Then, for all $\epsilon > 0$ there exists an $N = \max\{N_1, N_2\}$ such that for all $n \geq N$,

$$|x - y| \leq |x - x_n| + |x_n - y| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

by the Triangle Inequality. Hence, for all $\epsilon > 0$, $|x - y| < \epsilon$. Therefore, $x = y$. $\square$

Notation 17

We write $x = \lim_{n\to\infty} x_n$ or $x_n \to x$.

Example 18

Given the sequence $x_n = c\ \forall n$, $\lim_{n\to\infty} x_n = c$.

Proof: Let $\epsilon > 0$ and $M = 1$. Thus, for all $n \geq 1$,

$$|x_n - c| = |c - c| = 0 < \epsilon.$$

$\blacksquare$

Example 19

$\lim_{n\to\infty} \frac{1}{n} = 0$.

Proof: Let $\epsilon > 0$. Choose $M \in \mathbb{N}$ such that $M^{-1} > \epsilon^{-1}$. Hence, for all $n \geq M$, $\left|\frac{1}{n} - 0\right| = \frac{1}{n} \leq \frac{1}{M} \leq \epsilon$. $\blacksquare$