18.100A: Complete Lecture Notes

Lecture 9:

Limsup, Liminf, and the Bolzano-Weierstrass Theorem

Theorem 1 (Some Special Sequences)

What follows are some special sequences to have in our toolbox.

1. If $p > 0$, then $\lim_{n\to\infty} n^{-p} = 0$.
2. If $p > 0$ then $p^{\frac{1}{n}} = 1$.
3. $\lim_{n\to\infty} n^{\frac{1}{n}} = 1$.

Proof:

1. Let $\epsilon > 0$. Then, choose $M > (1/\epsilon)^{1/p}$. Hence, if $n \geq M$,
$$\left|\frac{1}{n^p} - 0\right| = \frac{1}{|n^p|} \leq \frac{1}{M^p} < \epsilon.$$
2. Suppose $p > 1$. Then, $p^{1/n} - 1 > 0$ which may be proven by induction. Furthermore, we have
$$\begin{aligned} p &= (1 + (p^{1/n} - 1))^n \\ &\geq 1 + n(p^{1/n} - 1). \end{aligned}$$
Therefore, $0 < p^{1/n} - 1 \leq \frac{p-1}{n}$. Hence, we may apply the Squeeze Theorem, obtaining $\lim_{n\to\infty} |p^{1/n} - 1| = 0$. If $p < 1$, then
$$\lim_{n\to\infty} p^{1/n} = \lim_{n\to\infty} \frac{1}{(1/p)^{1/n}} = \frac{1}{1} = 1.$$
Furthermore, if $p = 1$ then it is clear that $\lim_{n\to\infty} p^{1/n} = 1$. Hence, in all cases, the limit is 1.
3. Let $x_n = n^{1/n} - 1 \geq 0$. We want to show that $\lim_{n\to\infty} x_n = 0$, as this will imply the end result. Notice that
$$n = (1 + x_n)^n = \sum_{j=0}^{n} \binom{n}{j} x_n^j \geq \binom{n}{2} x_n^2 = \frac{n!}{2(n-2)!} \cdot x_n^2 = \frac{n(n-1)}{2} \cdot x_n^2.$$
Thus, for $n > 1$,
$$0 \leq x_n \leq \sqrt{\frac{2}{n-1}} \implies x_n \to 0.$$

□

Limsup/Liminf

Question 2. *Does a bounded sequence have a convergent subsequence?*

Definition 3 (Limsup/Liminf)

Let $\{x_n\}$ be a bounded sequence. We define, if the limits exist,

$$\limsup_{n\to\infty} x_n := \lim_{n\to\infty} \left(\sup\{x_k \mid k \geq n\}\right)$$
$$\liminf_{n\to\infty} x_n := \lim_{n\to\infty} \left(\inf\{x_k \mid k \geq n\}\right).$$

These are called the limit superior and limit inferior respectively.

We will now show that these limits always exist.

Theorem 4

Let $\{x_n\}$ be a bounded sequence, and let

$$a_n = \sup\{x_k \mid k \geq n\}$$
$$b_n = \inf\{x_k \mid k \geq n\}.$$

Then,

1. $\{a_n\}$ is monotone decreasing and bounded, and $\{b_n\}$ is monotone increasing and bounded.
2. $\liminf_{n\to\infty} x_n \leq \limsup_{n\to\infty} x_n$.

Proof

1. Since, $\forall \in \mathbb{N}$,
$$\{x_k \mid k \geq n+1\} \subseteq \{x_k \mid k \geq n\},$$
we have that $a_{n+1} = \sup\{x_k \mid k \geq n+1\} \leq \sup\{x_k \mid k \geq n\} = a_n$.
Similarly, $\forall n \in \mathbb{N}$, $b_{n+1} \geq b_n$. Given $\{x_n\}$ is a bounded sequence, $\exists B \geq 0$ such that $\forall n \in \mathbb{N}$,
$$-B \leq x_n \leq B.$$
Therefore, $\forall n \in \mathbb{N}$,
$$-B \leq b_n \leq a_n \leq B$$
which implies both sequences are bounded.
2. By the above equation, $\forall n \in \mathbb{N}$, $b_n \leq a_n \implies \liminf_{n\to\infty} x_n = \lim_{n\to\infty} b_n \leq \lim_{n\to\infty} a_n = \limsup_{n\to\infty} x_n$.

□

Let's consider a few examples.

Example 5

Let $x_n = (-1)^n$. Calculate the $\liminf$ and $\limsup$ of this sequence.

Proof: Notice that $\{(-1)^k \mid l \geq n\} = \{-1, 1\}$. Thus, the supremum of these sets is always 1 and the infimum is always -1. Therefore,

$$\limsup_{n\to\infty} x_n = 1 \quad \text{and} \quad \liminf_{n\to\infty} x_n = -1.$$

■

Example 6

Let $x_n = \frac{1}{n}$. Calculate the lim inf and lim sup of this sequence.

Proof: We may do this directly:

$$\sup\{1/k \mid k \geq n\} = \frac{1}{n} \to 0 \implies \limsup_{n\to\infty} x_n = 0.$$
$$\inf\{1/k \mid k \geq n\} = 0 \to 0 \implies \liminf_{n\to\infty} x_n = 0.$$

■

The limit inferior and the limit superior allow us to answer the question posed at the beginning of this section.

Theorem 7

Let $\{x_n\}$ be a bounded sequence. Then, there exists subsequences $\{x_{n_k}\}$ and $\{x_{m_k}\}$ such that

$$\lim_{k\to\infty} x_{n_k} = \limsup_{n\to\infty} x_n$$
$$\lim_{k\to\infty} x_{m_k} = \limsup_{n\to\infty} x_n.$$

Proof: Let $a_n = \sup\{x_k \mid k \geq n\}$. Then, $\exists n_1 \in \mathbb{N}$ such that $a_1 - 1 < x_{n_1} \leq a_1$. Now, $\exists n_2 > n_1$ such that

$$a_{n_1+1} - \frac{1}{2} < x_{n_2} \leq a_{n_1+1}$$

since

$$a_{n+1} = \sup\{x_k \mid k \geq n_1 + 1\}.$$

Similarly, $\exists n_3 > n_2$ such that

$$a_{n_2+1} - \frac{1}{3} < x_{n_3} \leq a_{n_2+1}.$$

Continuing in this way, we obtain a sequence of integers $n_1 < n_2 < n_3 < \dots$ such that

$$a_{n_k+1} - \frac{1}{k+1} < x_{n_k} \leq a_{n_k+1}.$$

Given $\lim_{k\to\infty} a_{n_k+1} = \limsup_{n\to\infty} x_n$, by the Squeeze Theorem,

$$\lim_{k\to\infty} x_{n_k} = \limsup_{n\to\infty} x_n.$$

The direction for the lim inf works out the same way so that portion of the proof is left to the reader. □

Theorem 8 (Bolzano-Weierstrass)

Every bounded sequence has a convergent subsequence.

Remark 9. *We may abbreviate the Bolzano-Weierstrass theorem to B-W.*

Proof: This follows immediately from the previous theorem, but is so important that it itself is a theorem. □

Notation 10

When it is clear, we may have the following notational shorthand: $\liminf_{n\to\infty} x_n := \liminf x_n$, and $\limsup_{n\to\infty} x_n := \limsup x_n$.

Theorem 11

Let $\{x_n\}$ be a bounded sequence. Then, $\{x_n\}$ converges if and only if $\liminf x_n = \limsup x_n$.

Proof ($\Longleftarrow$) Suppose $\liminf x_n = \limsup x_n$. Then, $\forall n \in \mathbb{N}$,

$$\inf\{x_k \mid k \geq n\} \leq x_n \leq \sup\{x_k \mid k \geq n\}.$$

By the Squeeze Theorem, since $\lim_{k\to\infty} \inf\{x_k \mid k \geq n\} = \lim_{k\to\infty} \sup\{x_k \mid k \geq n\}$ by assumption, we have

$$\lim_{n\to\infty} x_n = \liminf x_n = \limsup x_n.$$

Therefore, x_n converges.

($\Longrightarrow$) Let $x = \lim_{n\to\infty} x_n$. Therefore, every subsequence of $\{x_n\}$ converges to x, so $\liminf x_n = x$ and $\limsup x_n = x$ by a theorem we proved in Lecture 7. Hence, $\liminf x_n = \limsup x_n$. $\square$