[SQUEAKING]

[RUSTLING]

[CLICKING]

TOBIAS COLDING: OK, so last time we talked about, in particular, we talked about two things. We talked about the Cauchy Convergence Theorem. So the Cauchy Convergence Theorem says that any Cauchy sequence is convergent.

So this is one thing we talked quite a bit about last time. And we proved it. But we proved it assuming another theorem, so we showed it, the proof used the Bolzano-Weierstrass Theorem. Now, the Bolzano-Weierstrass Theorem-- so that's the following-- so this is the following theorem. Let's say that any bounded-- sorry, any bounded sequence, an, has a convergent subsequence. And let's call it ank. That was the Bolzano-Weierstrass Theorem.

So using this-- remember, the Cauchy Convergence Theorem, the proof then went like this, that if you take a Cauchy sequence, then it was easy to prove that it was bounded, then you use the Bolzano-Weierstrass Theorem to say that now the sequence is bounded-- it's a Cauchy sequence-- the sequence is bounded, so it has a convergent subsequence, and the limit-- you would then want to prove that the limit is actually a limit for the whole sequence. That's what we did, right?

But we didn't prove the Bolzano-Weierstrass Theorem. So let's try to prove it now, so proof of the Bolzano-Weierstrass. So let me try to explain the idea. So you have a sequence, a n, and a n is bounded. Let's just say it doesn't matter. But let's say that a n lie in the interval from 0 to 1. It could be any other bounded interval. But let's just say 0 to 1.

Then the idea is that you want to prove that an-- so the idea is that we want to find a subsequence, ank, and two other sequences, a-- sorry, bk and ck. And these two other sequences have the property that if you have here ank, then the b's lie below-- they lie below the a and k. And the c's lies above it.

So the sequence, the subsequence that we will find is going to be squeezed between two other sequences. So a and k is squeezed between two other sequences. And the thing about the other sequences is that they are much nicer. So why are they much nicer? The bk's are monotone increasing. And the ck are monotone decreasing.

So you have that. So the picture is that the b's are always moving in this direction. The c's here. This is the b's. As k gets larger, they're moving to the right. And the ck's are moving to the left. And these guys, a and k, they will be squeezed between these two.

So if you're looking at a later one, so this is like one. And now imagine that you take a later in the subsequence-- sorry, let me indicate it here, an, say, l, and l is now further out in the subsequence-- then this is going to be lying between bl and cl.

But these here, these here will lie-- the bl, because it's further out in subsequence, it will lie to the right of the-- so here will be bk. It would be bk would be to the left, and ck would be to the right of this, right?

So you see that these here are coming together. They're coming together like this. And the a, the subsequence is going to be squeezed in smaller and smaller interval. And that's going to force it to converge.

And the point here is that because this here is increasing, and they always lie between the ones that go below, this automatically means that the bk's will always be less than the first c. And these ck's will always be bigger than the first b.

And so this is a monotone sequence that go up, but it's bounded, so it has a limit. So bk is converging to something. The ck's are going down, but they're uniformly bounded from below. So they will also be converging, right?

But now they're coming closer together. They're coming closer and closer together. This is converging to that. And this is converging to this. So that's going to imply that b is equal to c. That's the idea. Yes, sir?

AUDIENCE: Do they need to be strictly monotonic?

TOBIAS COLDING: They will have-- sorry, the bk will be less or equal to bk plus 1. And the ck will be bigger or equal to ck plus 1. So it's not strict. It's inequality, like this.

AUDIENCE: [INAUDIBLE] you had a constant [INAUDIBLE] constant?

TOBIAS COLDING: Yeah that's right. But the thing is also that they are going to, that the difference-- so the ck is lying above the bk. So this here is positive. And we will prove that this here is bounded by 2. It was actually going to be equal to. We would actually prove that this here is equal to 1 minus, like this. And you see this here goes to 0. So you're right that if they didn't come together, we couldn't conclude things. But they will come together. This I will guarantee.

So how do we do this? That's the idea. How do we do this? Well, and again, I'm just doing 0 to 1. But the same argument works for any other interval. It will look simpler if I do it from 0 to 1, but it's not-- there's nothing essential I'm using about it.

So here is your starting interval from 0 to 1. We want to construct a subsequence. And not only do we want to construct a subsequence, but we also want to construct the sequence bk and the sequence ck, right?

So let me define b1. b1 is just going to be 0. c1 is going to be 1. And then an1-- that's the subsequence, the first element is subsequence. I just take it to be the first element in sequence. So now of course, we have trivially that we have that b1, which is 0, this is less than or equal to a1, because it lie in this interval, less than or equal to c1, which is 1.

Now, then you divide it into two intervals here. So the second step is that you take the interval from 0 to 1. You divide it into two halves. So you have between 0 and 1/2 and 1/2 and 1. The whole sequence lie in this interval. So this means that there are either infinitely many here or infinitely many there, right? Let's say that infinitely many of the a n lie in this interval. So then we're going to forward. We were going to just focus on this interval and on subintervals of this.

So we have defined a1. Now we want to define-- and b1 and c1. So now I can assume that infinitely many a n's lie in this interval, right? So then we take an2-- the second element in a subsequence, we let it be the next element after a1 that is in this smaller interval. So this here is the second element in the original sequence that lies in the interval from 0 to 1/2. OK?

That's an2, right? We don't want to have a1 again. We need another element, because otherwise it's not going to be a subsequence. You can't pick the A1. I mean, it could be the same value. But it will have to come from a different subscript.

Now, b2 is then just going to be the left endpoint of this interval. So this is again 0. And c2 is going to be the right endpoint of this interval we're now focusing on. So that's 1/2.

Now, so now to the next-- so now, how do we construct the third element in the subsequence and the corresponding b's and c's? The thing is that-- and here you have to be a little bit careful, because now, we assumed that there was infinitely many elements in the sequence that lie in this smaller interval. You are going to only-- from here on out, you're going to only look at those infinitely many. You're not going to care about, you know, so this may be the fifth, the seventh, the 15th element in the original sequence. You're not going to worry about what lies here.

And so now you divide this one here into two halves. This interval between 0 and 1/2, you divide it into halves. And so now, imagine you have then the picture here. Here you have 0. Here you have a 1/2. And then you divide it into this integral from 0 to 1/2. You divide it into two equally large intervals.

So now, we're just focusing on the infinitely many of the original sequences that lie here. But of those infinitely many, either infinitely many lie here, or infinitely many lie there. Let's say that there's only finitely many here, say, but there are infinitely many here.

So if there's infinitely many in this interval here, then from here on out, you would only focus on those infinitely many. And the next element would be-- we know that by the next element would be the next element after the one you pick to be the second element in the subsequence that would lie in this subsequence.

So we define a3 is now the next element. So an3, this is now this element that lies in this interval here. And then b3 is then going to be 1 over 4. And b-- sorry, and c3 is going to be the right endpoint of this interval, 1/4-- 1/2, sorry.

And then you continue. Then you continue this way. So let me just make sure that it's totally clear. Let me just go over how we did it, so just because I want to be careful that once you-- so we started with the interval from 0 to 1. The first element you just pick in the subsequence, that was just the first element in the original sequence. And the b1 was just the left endpoint here, so that was 0. c1 was the right endpoint. That was 1.

Now you divide it into 2. And one of the two halves-- potentially, of course, both of the two halves could have infinitely many elements in the original sequence. You just focus on one of the two halves that have infinite elements. If they're both, fine, but you just take one of them.

And so from here on out, we're just going to focus on those infinitely many. And then you're going to pick the next one after the a1 that lies in that interval you're focusing on. That's the second element in the subsequence. And the b's will always be the left endpoint of the interval. The c's will be the right endpoint of the interval.

Then when you do the third element, then you have to remember that you were looking at infinitely many. And so now it's like you're looking at-- you divide this interval into two. And now either this one here or that one here has infinitely many. And you pick whatever it is. I mean, I guess we assume it was this interval.

And then the next element after the two first you picked in the original sequence, the next element that lies in this, that's going to be your third element of the sequence. And then still, the b's will be then of this interval you're focusing on would be the left endpoint and the c, the right endpoint.

If you do that, you see each time you're halving it. So it means-- well, first of all, it means that the b's are going to be increasing, because it's always by each time you divide into a half, and it's always-- but the interval you focus on previously you divide into half, like this. And then either you pick either this point or that point. And this was the previous one. So the b's will always be increasing. Likewise, the c's will be a decreasing sequence.

And you also see that the difference between ck-- say, ck plus 1 minus bk plus 1-- that the difference here, but the difference is going to be like half of the length of the previous interval. So the difference here is going to be 1/2-- it's actually going to be 1/2 and then bck minus bk. So you're getting this form here.

And now if you just use that you're starting with-- if you're just using that there are [INAUDIBLE], so you get this recursive formula here. And you also have that c1 minus b1. This is the length of the original interval, because b1 was 0, c1 was 1. So this is 1.

And so you're just putting this in. And so you get the formula that ck minus bk is equal to 2 to the 1 minus k. So you're getting this formula here. This just comes from this recursive formula and how this starts. So this gives you this.

And so you see that the length of this interval becomes smaller and smaller. Again, the sequence bk is increasing. The ck is decreasing. So this means that this here is converging to some b. This is converging to some c. And because the difference between these goes to 0, then the b must be equal to c.

And so this will prove that the limit that you construct, because this is going to be squeezed, you will have this property that if you're looking at ak and even-- sorry, ank and even anl, the addition here, this here will, by construction, will lie between these two.

But all the later ones will also lie in this interval. So this here will also lie in this interval, ck, as long as l here is bigger or equal to k. And so you see that this thing here, this thing here will-- because this here is converging to b, and actually now, we prove that this converges to the same number, b, so this means that these guys here that is squeezed between these two other sequence will also converge to b. So that's a Bolzano-Weierstrass Theorem.

OK, another key concept in this class is the concept of a series. And so this is the concept of a series, oops. And so a series is like as follows, that suppose you have-- so let an be a sequence. From a sequence, you can form another sequence. And you can form another sequence in the following way, that s-- so we form another sequence.

And that is that s1. And we call this other sequence sn, another sequence, sn. And the sequence is defined as follows, that s1 is a1, s2 is a1 plus a2, s3 is a1 plus a2 plus a3. s4 is where you're summing the first four elements in the original sequence. And then you write-- sorry, this is the first four element. And then you write the n-th element. You write it just as a sum where i equals to 1 to n of the ai's.

So if you have a sequence, then you can form another sequence. And this we denote by s. And this here is-- so this here, other sequence, if a sequence is of this form, then you call it a series, so sn here. Sorry. But sn is just a sequence. This here is said to be a series.

Now, let's go to the most-- so I'm sure probably everybody have heard about this kind of one of the famous Greek paradoxes is this. I mean, they have various different forms. This is one form of it is with Achilles and the turtle, where the turtle is getting a head start of Achilles.

And the paradox is that because the turtle have a head start, then Achilles can never catch up with the turtle, even though the turtle is moving incredibly slowly, because in order to catch up with the turtle, then Achilles first have to go half of the distance that the turtle have gone. And then once you've done that, you have to go half of the distance that the turtle now has traveled, et cetera. And so this goes on infinitely long, and therefore, he can never catch up with the turtle. There's various other version of this.

And that's exactly a series. And this is also like the most important series is what's called the geometric series, so the geometric series. So this is the most important series. And it's the most important because it's kind of a non-trivial series that can be used to determine whether other series or whether for other sums, whether this sequence sn, whether it converges or not.

So the geometric series is the following, that c-- you take some number, c-- and then you are summing this number, but to a power. So you're looking at where you have some power, say, from 0 to n. So this means that-- so this is sn. So this means that s serum, this is just where you have a-- so c to the power 0 by definition is 1. And s1 is then when you're summing c to the power 0 plus c to the power 1. So that's just 1 plus c.

And s2 is then c0 plus c1 plus c squared. So that's just 1 plus c plus c squared et cetera. So that's a geometric series where it's just a number. And each time you multiply that number by itself the n time, and then you're adding it to the previous. So that's a geometric series.

And the reason why this is important is that this is very useful to determine whether other series are convergent or not because you could compare it, you can compare other series with the geometric series. And the geometric series is easier to determine whether it is convergent or not. And so this is because of the following little calculation here.

Let's suppose I'm looking at some summing here from 0 to n of c to the power i. And now I multiply this with c. So then this here would just be where I'm summing here c-- if I multiply by c, I can move that c inside. And then the first time it's really right-- so you can write this thing here. So let me just write it out.

So this is c. And then it's 1 plus c plus c squared plus c cubed, et cetera, like that, up to cn. And if you multiply by c, then, of course, this is just c plus c squared plus c cubed up to c to the n plus 1.

And so this means that if you take this thing here, so if you're looking at the sum from 0 to n of ci, and now you multiply by 1 minus c, then this is the same as just 1 times this. So let me just write it out. This just gives you the original series. So this gives you this up to cn. That's when we multiply the one n. When we multiply it by minus c, then we get this thing here with a minus sign. So we're getting minus c. So I can write it like this. Like that, right?

But now you see that there's a lot of cancelation. This thing here cancel with this here. The next one would cancel with this. This here would cancel with the second last. And so you are only left with this one here and that one here. But this one here comes in with a minus sign. So this is 1 minus c to the n plus 1.

So we have this little formula. And this is a formula, say, that if you take 1 minus c and you sum from i equal to 0 to n of ci-- this is what we did here-- then you're getting 1 minus c to the n plus 1. OK.

So now, of course, if c is 1-- let's wait with figuring that out. But assume that c here is not equal to 1. Then I can write this here as that this sum here, i equal to 0 to n ci, I can just divide here by this here on both sides. So I get this thing here, 1 minus c. So I have this formula here, assuming that c is not equal to 1, so I can divide by 1 minus c.

And now you see that-- so now from this here, we can determine-- so we can prove a little theorem. So that the geometric series-- so this is geometric series convergence. Remember that this thing here, so a series here-- so maybe I should just emphasize that over here.

So if you write down a series, so again, a series is always that you're starting with a sequence like this. And then you form this other sequence, sn, where you're summing the first n-- I guess in this case, I'm starting from 0. So then it would be n plus 1 elements, like this.

And sometimes one also just write for the series, one write it as this, where one write like this. But really, what you're thinking about is this sequence here. But you, with a slight abuse of notation, you write this here for series. But what you're thinking about is this sequence.

And so a series is convergent. It just means that the sequence sn is convergent. So it just means that this sequence, sn, is convergent. And the sum and the limit of the series-- this is just the limit, is the limit of the sn's.

That is usually denoted by this, by this infinite sum, a. Sorry, this was i, ai, like that. Right. So you see there's a little bit of abuse of notation, that one talk about a series and one typically write it like this. What one really means is that one thing about this sequence here. And then if it's convergent, this just means that these sn converge.

And the limit, but the limit you would also write as this. So in a way, this symbol here is used for two different things. It's used for the series. And if the series is convergent, it's also used for the limit. This is like standard.

OK, so now let's come back to the geometric series. So again, the geometric series were that you take some c, and then you form this series here, ci, i equal to 0 to infinity. It's, of course, important where you start. I mean, the convergence is not important. But what the limit is depends on where you start.

So this is a geometric series. And we have that if c here-- so we have now the following, that the geometric series is convergent if c here in absolute value is less than 1 and divergent otherwise, if the absolute value of c is bigger than 1.

So this is the claim that-- and so this is a claim. And in fact, so one more thing is that-- so moreover, if c is less than 1, then the limit-- so then the claim is that it's convergent-- then the limit is 1 minus c. This is the limit. And so we write this here as this infinite sum.

Now, let's just try to prove this. So let me first do-- and we're almost there already. So the thing is that-- so let's first look at the case. Assume c in absolute value is strictly less than 1. But then we have already proven that sn is-- so sn, this here was the sum from i equal to 0 to n of ci. And we have already proven that this is 1 minus cn plus 1 over 1 minus c.

And this makes sense. Norm is strictly less than 1. So in particular, c is not 1, right? So this here is fine.

But now you see that if you take something with norm strictly less than 1, then just because cn plus 1, the norm of this here, but if you take something and you multiply it by another-- so if you take two numbers, a and b, and you take the absolute value of the product, that's the same as the product of the absolute values.

So this means that if you take c to the power n plus 1-- so if you take c and you multiply it by itself n plus 1 time, this is the same as the absolute value times [INAUDIBLE] raised to the power n plus 1 right. And so now you see that because this c has absolute value less than 1, strictly less than 1, then as n goes to infinity, this here converges to 0.

So this means that as n goes to infinity, this thing here might converge. This here goes to 0. So just by the algebraic rules for limits, this thing here is going to converge to 1 minus c over 1, like that. So you see that in the case where c in absolute value is strictly less than 1, then we have now proven that the geometric series is convergent. And in fact, we have even, which is really nice, and unusual for a series. I mean, usually you cannot actually determine what the limit is. But in this case, we could.

But we still need to figure out why is it not convergent if the absolute value of c is bigger or equal to 1. And so here's an easy thing. I mean, but you could also-- yeah, so here's an easy thing, that if you're looking at c-- sorry, sn plus 1 minus sn, where sn plus 1 or sn say, this here was just c to the power 0 up to c to the power n.

Sn plus 1 is where you do the same stuff. We're adding these guys. But then you have to add one more, right? So this means that this difference here is really c to the n plus 1, right? But now I could look at the absolute value of this difference. This is this. But this is the same, again, as c to the power n plus 1.

But you see, if c here is the absolute value is bigger or equal to 1, then this thing here-- oops-- then this thing here, so if the absolute value here is bigger or equal to 1, then this thing here must be bigger or equal to 1. Remember that if a sequence is convergent, if a sequence is convergent, then it is a Cauchy sequence. It's a Cauchy sequence that if you go sufficiently far out, then from there on and out, everything bunch very, very close together.

But you see here, not even two consecutive is bunching together, because independent on how far, how large n is, the difference is always at least 1. So this is not even a Cauchy-- I mean, well, it's actually equivalent, of course, but it's not a Cauchy sequence. It's failed. Like the most basic thing with consecutive ones does not even lie close together. So this proves this theorem about the geometric convergence, the geometric series convergence.

So this is the most important example of a sequence. Maybe the second most important is what's called the harmonic series, so harmonic series. But the harmonic series is unfortunately not a series that is convergent.

So the harmonic series is the following. So again, I'm writing it-- I'm using this abusive notation that I'm writing the series as like an infinite sum. But this infinite sum, but it's not a convergent series, so the infinite sum doesn't really make sense. So here I have to start because I'm dividing by n. So I'm starting by 1 with 1. So this is called the harmonic series. OK.

And so for series, there's a bunch of things, a bunch of tests that are useful to determine whether a series is convergent or not. And we will talk about these tests. I'll talk a little bit about later in this lecture. I'll talk about some of the tests, but not all of them. And we will later talk about more.

The test that is particularly useful in this case, it's what's called the integral test. But I won't talk about that today. But instead, I will show you a very simple proof of why this here is not convergent.

So here's a theorem. The harmonic series is a divergence. So it's not convergent. It's divergent. And so in fact, what we will prove is the following, that we will show that if you're looking at-- so n is an integer. If you're looking at-- so this is like-- so this is only some of the-- you only hit, for this here, you only hit some of the elements in the sequence. So this is where I'm summing again from i equal to 1 to 2 to the n minus 1, n is an integer, of 1 over i.

So I want to prove that these guys here, if the original sequence had been convergent, then of course, any subsequence would also be convergent. And we're going to prove that this thing here is bigger or equal to n over 2. So I claim that we have this. So this is a claim that you have this. But if you have this thing here, you see, that then this subsequence here is unbounded. So of course, the original harmonic series is not convergent.

So let's try to prove this. Again, this is just like a simple proof of that-- very simple-minded proof-- that the harmonic series is divergent. But there are more. Once we learn about the integral test and so on, there are other way better ways of doing it.

So let me try to prove this. So I want to prove the claim, proof of the claim. So I'm going to prove it, right? So I just proved. So I want to establish this for all n. And I'm going to prove it first when n is equal to 1. So let's show this for n equal to 1.

So for n equal to 1, well then, the claim here means that-- so 2 to the 1, that's true. So this is s1, right, because 2 to the 1 is 2 minus 1. So this is 1. So I claim that.

So I want to prove that s1 is bigger or equal to 1/2. [INAUDIBLE] to show. But of course, s1 here, s1 is just 1 over 1. So that's definitely fine. So this is-- so we want to prove the claim. And the claim is OK. So claim is OK for n equal to 1.

So now I want to prove it. So I want to assume the claim is correct for n. And we want to prove it for, want to show the claim for the next one. And so I have that. So we assume that s2n minus 1, that this is bigger than n over 2.

And then I want to look at s2n plus 1 minus 1. OK, how can I write this thing here? So I'm summing, right? This here is a sum of the 1 over i, where I do it up to this thing here. But I can think about that as the sum of the one where I do up to this here. And then I have to sum the rest also. So this is i equal to 2n, to 2n plus 1, 2n plus 1 minus 1, like this, of 1 over i.

But now you see, I want a bounded from below. Fine. So if you continue this line, then by assumption, this thing here is bounded by n over 2. This thing here, I'm summing 2n numbers. From here to here, there are 2n numbers. And so it's 2n numbers.

And if I take the smallest of those two numbers, if I take the smallest of these guys and I multiply it by 2n, then that's definitely a lower bound. So let's try to do this. So I say 2n, 2 to the n-- and then the smallest, but the smallest is the last one. So the smallest is 1 over 2 to the n plus 1 minus 1.

But this thing here is, of course, obviously bigger than-- so n over 2, this thing here-- well, this is true, then. But this thing here, if I divide by a larger number and the larger number is just this here, then I get something even smaller. So I would have 1 over 2 to the n plus 1.

But now you see that this thing here cancel with these guys here. And so you're getting that this is n over 2 plus 1/2. And so this is n plus 1 over 2. And that's exactly the claim for n plus 1. So we're proven that if it was true for n, then it's also true for n plus 1. And we remember to check that the beginning was also OK.

So that's a harmonic series. So you see that the harmonic series is divergent. Now, the harmonic series is, of course, an example of something else, namely that if you're looking at-- so when you're looking at a series, the first thing you look at typically is-- OK, so let me just have a side here.

So if you give me a series-- it doesn't really matter where. I mean, it doesn't matter very much where it's-- for whether a series is convergent, it doesn't matter where you start, because this is just adding something to it. It will matter where you start for the limit, but not whether or not the series is convergent.

But if you give me a limit, the first thing I always check is that this-- so first, so for a series, always check if an goes to 0 or not. And why? Because if you give me a series, the question about a series is always whether it's converging or not. And again, just like we did, if you're looking at sn, say, minus sn minus 1, this is where I'm looking at two consecutive things.

But if I'm looking at this difference here, the difference here, this is where I'm summing up to the a, n minus 1. Here I'm summing up to the an. So the difference here is an. And if the sequence-- just like we saw, if the sequence of the series is convergent, it means that this sequence here is convergent. But it means that it has to be a Cauchy sequence, that things have to bunch together, but that two consecutive bunch together is determined whether an goes to 0 or not.

So if an does not go to 0, you're already done, right? There's no reason to continue. It's already clear that if an does not go to 0, you can just say it doesn't converge, because an doesn't go to 0. So there's no reason to start with any kind of test or anything, any sort of good ideas.

So that was just a side remark. The other thing is that for the harmonic series is-- so this, of course, failed this test. But it also is an example of something where you're taking-- so suppose you take a-- so you have a sequence, an. And then you're forming the series, this series here.

Now, if all of a n's-- you post that all of the a n's are non-negative, then if you're looking at sn plus 1, you can think about sn plus 1 as where you're summing the first n elements, and then you're adding the n plus 1 element. So this here you can think about as sn plus an plus 1.

And so of course, if all of these elements is non-negative, this means that this has non-negative. So this thing here is bigger than sn. So you see that if all of the element, an, is non-negative, then this sequence, sn, is monotone. So in this case here, if you have this, this imply that this sn is monotone increasing. And so now to say whether this, if you have this, to determine whether this series here is convergent or not, it's just a matter of proving whether to determine whether this sequence is bounded or not. So you just need to find a bound.

So this naturally leads to looking at what's called absolute convergence. So this is something stronger than convergence. And for many things-- for many, although not all-- there is a particular test that deals with it where you don't have both things or you [INAUDIBLE] have a stronger property. So for many things you have absolute convergence.

And so let me try to-- so absolute convergence. So again, you have a series here. This series here is said to be absolute convergence or converge absolutely-- so converge absolutely-- if the series where you slap absolute values on them, if this thing here, converge.

So we say that this here converge absolutely if this here converge. Now, so I claim that absolute convergence is stronger than convergence. So absolute convergence implies convergence. So this is a theorem.

But it's not the other way around. So if you have what's called the-- so here, an example. And we haven't yet proved that it's converging, but we'll do that in some later lecture. So here's an example. If you're looking at the series minus 1 to the power n divided by n, n equal to 1 to infinity, let's say you see if this had just had been 1, then it had been the harmonic series.

This here series is also kind of like a famous series. It's called the alternating harmonic series. And it's alternating. Alternating means that the sign of it is alternating. So the sign of this an element, it keeps changing from a-- in this case, the first one is negative. The next one is positive. So go back and forth between around 0, so harmonic series.

And this series is convergent, but not absolutely convergent, because if you slap absolute value sign, you get the harmonic series. And we already know that this is not convergent, but not absolutely convergent. So this is the alternating harmonic series.

Now, let's try to prove that absolute convergence implies convergence. And so I'm going to look at-- so I take two series. So I have sn. That's my original series, ai, i equal to 0, to n, like that. And then I take the one where I snap absolute value sign and I denote it with a bar, so i equal to 0 to n of ai.

And then I make a little observation. And the observation is that if I take-- if I'm looking at sn and sm, so just think about m here is larger than n, then you can think about this thing here, as this thing here is just where you're summing from n plus 1 n up to an. That's this difference, right?

But if you take absolute value on this, absolute value on that, well, that's of course lesser or equal to where you're taking absolute value on each individual one, like that. But this thing here is just-- this thing here is just for the other series, this series. It's the same difference.

And here, actually, I don't even need to take absolute value, but I could take absolute value. But this is always [INAUDIBLE]. And so you see that-- so you make this little observation. But now we know that if it is absolutely convergent, then this sequence here converge. So this means that in particular, it's equivalent to saying that it's a Cauchy sequence. And so this means that since if you are sufficiently far out, these things have bunched together, epsilon together if you are bigger than some capital N.

But now, this inequality say that this thing here bunch together. So you see that if this is convergent, then it's a Cauchy sequence. But this means that this here is a Cauchy sequence. So it means that the original series was convergent.

And so now, let's look at one more example. And then we'll talk about some of the tests. So let's look at-- so we saw that the harmonic series is divergent. So we saw that the harmonic series is divergent. But what about-- I mean, what about-- so there's another thing you could do.

So maybe the most important series is the geometric series. After that, you can debate a little bit which series is the most important. But this here is some other candidates for that. So I'm looking at now the sum here, n equal to 0, to infinity. And I'm looking at 1 over n squared. But you could do it more generally where you raise it to some power. If that power was 1, you got the alternating. You got the harmonic series. But now I'm just looking at a specific power, 2. OK.

And so I claim that this series here, this here is-- I mean, this one doesn't really have a name, but it is a key series. And so I claim that this here is convergent.

Now, you observe that the an, in this case, this is just 1 over n squared. And these here are non-negative. So all we need to do is-- so we just need to prove. So when the a n's are non-negative, all you need to prove is that the sequence-- you just need to prove, need to show that the sn, which is where you're summing here, i equal to-- or you can't, you shouldn't sum from 0, sum from 1, i equal to 1 to n-- that we will need to prove that these here are bounded. This is because the a n's are non-negative.

So now the idea here is that-- I mean, again, this is not necessarily how I would do it. But the advantage with this here is that it doesn't use anything fancy. I mean, it's not like the way later on you do it, you will use the integral test. But we haven't talked about this integral test. So you can do it very simple minded.

And so the claim here is that you want to compare. So we want to make a foreign claim that if you're looking at s to the 2n plus 1-- sorry, to the 2n, so it's just similar to what we did before. If we want to prove that this thing here for this series, that this is less than the sum here from i equal to 0 to n of 1/2 to the power, i, so we'll put n to n minus 1. So we claim this thing here.

Now, you see that-- so we claim that we have this inequality for all n. But this thing here, we already know that this here is a geometric series with c equal to 1/2, so it's convergent. And you see this thing here is a subsequence. But it's a subsequence because this is convergent. This here is bounded by something fixed.

So this means that this subsequent here is bounded. But the original series was a monotone increasing sequence. So if a subsequence is bounded, then the original sequence is bounded and so convergent. So we just need to prove-- we just need to give a bound, like this.

And the bound here you get is by we already determined that this here is less than or equal to 1 over 1 minus c. c is 1/2. So this is like this. So this is 1/2. So this whole thing here is 2. So this here would be bounded by 2, right?

So now we want to prove. So we want to prove this claim here. So if you can prove this claim, then we have that the series, this series here is convergent. OK. So now let's try to do that.

So when n is equal to 1, so we do it again in the same way, which is called by induction. We do it for, first of all, for n equal to 1. For n equal to 1, it is a claim that s-- if n is equal to 1, this is true, minus 1. That's 1. So it's a claim that this thing here is bigger equals-- it's less than or equal to.

And now when n is equal to 1, you're just summing over one thing. It's just one thing. And it is 1/2 to the power 0. And that's 1. So for n equal to 1, this is what we need to show.

But now, let's see for n equal to 1, well, this here is indeed 1. So that's OK. So this is OK. So now assume that s to the power 2n minus 1 is bounded by this i equal to 0 to n minus 1 of 1/2 to the power i. So assume this. And we want to prove the same thing, but for n plus 1, so want to show the same inequality for n plus 1. OK.

So let me do that. So I'm looking at s2n plus 1 minus 1. So s2n plus 1 minus 1 I can write as where I'm summing the first 2n minus 1. And then I'm summing, and then plus, and then where I'm summing from i equal to 2n to up to 2n plus 1 minus 1. And then it's 1 over i squared.

Now, in this case, remember before we wanted to prove that something was divergent. So we wanted to give a lower bound. In this case, we want to give an upper bound. So when we do this, we think about what is the-- instead of what is the smallest, we think about what is the largest of these numbers. But that's the first one.

And there are, again, just like before, there are 2n many of them. So this thing here is less than or equal to s2n minus 1. Sorry, like this. And then there are 2n many. They're all non-negative. And the first one is the largest, so 1 over 2n squared, like this, right.

And so you see that this thing here is-- so now we can fill in the assumption. We are assuming that this here was less than this. This is sum from i equal to 0 to n minus 1 of 1/2 to the power i. And this thing here, but the square kills that one and you're still left with 1 over 2n.

But you see, this is, of course, exactly the next one in the series. So this is i equal to 0 to n of 1/2 to the power i, again. And that's, indeed, this is like the induction step, right? This is what's called the induction step. So it's true also. So if it's true for n, the claim, then it's also true for n plus 1.

So now let me-- I have just four minutes left. There's a little bit more, and I already posted the lecture notes. And there's a bit more here I want to talk about. So I want to talk about some of the tests. But we'll talk about that more next time, also.

So to determine whether or not a series is convergent, converge, then there are a number of tests. And then of course, the key point is to use the right test. So I will just talk about-- so I just mentioned some of the tests. But we won't anyway go through all of them now. So maybe I can't go through any of them.

So there's a comparison test. There's a ratio test. There's root test. And there's also the one I talked about several times, the integral test. And then there's even one called the alternate series test.

Now, maybe I can just talk about the comparison test. That's, in a way, the simplest of them. And so the comparison test is that you're looking at two. So you have these two series here. And you have this other series here.

And the comparison tests say that if you have an equal to 0, lesser equal to bn-- so if you have this end, and this here is here, converge, then the corresponding thing with an will also converge. And this is the simplest version of the comparison test. And this is, in a way, used in two ways.

So often you have a series. And maybe it could be a super complicated series. And but you see some kind of path. And so you can compare it with a much simpler series that perhaps you already know whether it's converging. And so if you can compare it with something else and you know that that converge, then you conclude that the an series is convergent.

It's also sometimes used that you want to determine whether this series is convergent. But actually, you can easily compare it with some other series where you have this relationship here, and this one you already know is divergent. So it's used in both ways. It's used to both determine that a series is convergent. And it's also used to determine that the series is divergent. So there's a little bit more about tests in the lecture notes. But we will, anyway, talk about it more next time. Yeah?

AUDIENCE: I have a question about the intuition of this [INAUDIBLE].

TOBIAS COLDING: Right. So that's right. So that sort of thing is-- I mean, this last one shows you that the geometric series is one that come up often as a kind of comparison, in a way. How you get it is different for different people. Obviously you don't start with the claim. You're looking at the first few things and you see a pattern, basically.

But it's not-- I mean, this is not the way you want to really-- later on, I would never prove any kind of convergence like this. There's a way of comparing it with the-- so the idea is, let me just tell you. But of course, the lecture is over.

So the integral test is that you're looking at-- so you're looking at a series. And if you have a series of non-negative numbers, you can think about that the series is like the area below. If you think about this here as a1, and this is maybe a2, then you can think about the sum as the sum of the areas, like this. And so the integral of testers that you're comparing it with some curve and the area that lies below it, then it's much more clear.

The other one is like where you have to do it case by case. Here there's a kind of general idea. So it's nice, in this particular case. I'm not sure. You know what I mean? I mean, you have to look at the first few things and see if you see a pattern. And of course, you don't start with a claim. That's like the end, when you actually have a claim, then you already know what's going on.