[SQUEAKING]

[RUSTLING]

[CLICKING]

PROFESSOR: So last time, just quickly recall, we looked at a function on some interval into R. We looked at a partition, P. So here, we have the interval from a to b. Then we have a partition, x0, x1, up to, say, xn.

We looked at Mi, which was the sup, on this little interval here from xi minus 1 to xi of f. We looked at little mi, which was the inf. f here was bounded again. And inf on the same interval, xi minus 1 to xi.

And then we looked at the upper and the lower sum. So we looked at the upper and lower sum with respect to this partition. So this here was the sum of Mi and then delta xi and the lower sum for this partition, which was the sum of this smaller mi's delta xi, where the delta xi was the length of these little intervals here, like that.

And then it was Riemann integrable. Then we looked at the lower and the upper Riemann integral. So we looked at the upper and lower Riemann integral. This here was now the inf of these upper integrals over all partitions. And the lower one, that was the sup of the lower sums over all partitions.

And then we said that f is Riemann integrable if those two numbers are the same, if the upper one here, which is always bigger or equal to the lower one, if that is actually equal to the lower one, the lower Riemann integral. And if this is the case-- so this here is-- it's Riemann integrable if and only if these two numbers are the same. And in that case, we wrote-- for this common number, we wrote just the-- we wrote it as the Riemann integral.

So now, there was this parallel to things we have done before. There was these algebraic rules that we talked about at the very end. So what were those? If you take a function that is Riemann integrable, and you take c is a constant, then the product here is also Riemann integrable. And the integral of the product is just-- product function is just the product of c with the Riemann integral of f.

So that was 1. 2 is that if you have a second function, so in addition to f, you also have a g, and they're both Riemann integrable, then the sum here is also Riemann integrable. And the integral of the sum is equal to-- from a to b-- is equal to the sum of the Riemann integral. Let's prove this one here. I won't prove the first one. But let's prove the second one. So this is 2.

Now, to prove this, let me remind you about one of the lemmas that we had last time. We were talking about that this was a useful lemma to determine whether a function was Riemann integrable. So recall this from last time, that you have the following lemma, that f, again, is bounded. So f is Riemann integrable if and only if, for all epsilon greater than 0, there exists a partition of this interval here such that the upper sum, which is always bigger than the lower sum, this one here, minus the lower sum-- so this number is always positive.

But if, for all epsilon, you can make this difference here smaller, then you can find a partition so that you have that. You proved that last time. So now, let's go to the proof of this. So proof of 2. So given epsilon greater than 0, since f here is Riemann integrable, then there exists a partition, P1, such that the upper sum minus the lower sum is less than epsilon over 2.

This is because the function f was Riemann integrable. And likewise-- likewise, since g is Riemann integrable, so there exists a second partition, P2, so that, for g, you have the same thing-- for g and P2. Sorry, this is g. This upper sum minus the lower sum for g with respect to this second partition, this here, is less than epsilon over 2.

So now you let P here-- so now let P be the partition that has all the dividing points of P1, together with all the dividing points of P2. So it has the partitions that are in either one of the two, or both.

And so now we know that-- so this just means that-- so P here is a refinement of both P1 and P2. So this means that actually, when you're looking at this, if you're looking at this upper sum minus this lower sum, remember that if you go to a refinement, then this here goes down. And this here goes up.

So it means that this difference here would be less than or equal to what it is for P1. And this was, of course, less than epsilon over 2. That was how it was chosen. And similarly, if you're looking at the upper sum for g with respect to P, P here is a refinement of P2 also. So this means that you have that this difference here is less or equal to the upper sum of g with respect to P2 minus this lower sum with respect to P2. And that's less than epsilon over 2.

But this means now that if you now take the-- so now, if you take-- so now, if you take an interval here of-- this is from xi minus 1 to xi. If you're looking at this sum here-- you have to be a little bit careful here. But if you're looking at the sum on this integral, the sum here is less or equal to the sup of what f is-- this is on this interval here.

On this interval, this function here is less than the sup of f plus the sup of g on the same interval. And it's, of course, bigger or equal to what-- it is the inf of f plus the inf of g on that same interval.

Why do I say you have to be a little bit careful is that the inf of the sum could be strictly larger than this. And it could be strictly less than that. But you certainly have this statement. Just thinking about, if f and g was the same up to a sign, if g was like minus f, then the sum would be dead 0. But one of the functions may be negative on this interval, and the other one was positive. So you see, for the sum here, could be dead 0. But these two numbers, one would be positive. The other would be negative. But you certainly have this inequality.

And so, of course, this implies that on this interval, the sup of the sum here is less than the sup of the sum of the sups-- sum of the sups on the same intervals-- and likewise for the inf.

So you have this thing here. And you have that this thing here is clearly bigger or equal to the inf of the same function, which is the sum of the two, on this interval here. And that is, of course, bigger than this sum here.

So this here, this number here-- you see, this here is like Mi for the function f. This is what this number is. And this here is like-- this number here is like capital Mi for the function g. And this here-- this here is like little-- this here is like little mi for the function f. And this number here is little mi for the function g.

So what we have here is that mi for f plus-- sorry, let me write this-- let me write it over here. So what we have is that we have little mi for the function f plus little mi for the function g is less than or equal to little mi for the sum, which is, of course, less than the sup of the same sum on the same interval. So this is this.

And that's less or equal to the sup of f on that interval plus this here of g. So you have that inequality. But now you see that this means, of course, that the contribution-- so if you're looking at this upper sum for this-- this upper sum for the sum of these two functions with respect to P, this here, the contribution you get is, for the upper, you get this. And for the lower, you would get-- you'd get this here times the length of the interval, right?

So this is-- so again, so what you get on that little interval, the contribution you get is like Mi f plus g times the length of the interval-- this is this-- minus Mi f plus g times the length of the interval. This is the contribution that you get here from this interval-- from this interval, right?

And because this number here is less than that, then this here is less than Mi f plus Mi g times delta xi minus-- and then this thing here, you see, this spreads between these two here. Bounds are spread between those two. So this is less than mi f plus little mi g times delta xi.

And now you're summing over everything. So this means that this thing here is-- so this-- and if you just split this here, this sum here, you can just split that into two parts, the Mi f times delta xi minus little mi f times delta xi, and then plus capital Mi g delta xi minus little mi g delta xi.

So when you're summing these up, you get a U f, P. That's from this here. When you're summing these up, U f, P minus, sorry, L f, P. When you're summing this up, then you're getting U g, P minus L g, P.

And so what you get here is-- again, when you're summing it up, this here gives you-- when you sum these up, then you get this. When you sum this here up, you get that. When you sum this up, you get that. So summing everything up, you get U f plus g, comma, this partition, minus L f plus g with respect to this partition, is less or equal to U f with respect to this partition, minus L f with respect to partition P, plus U g, P, minus L g, P. So you're getting that.

And this thing here, it was a refinement of the partition P1. So this spread here is less than what it was for P1. So this here is less than epsilon over 2. And likewise, this here is a refinement of the partition P2. So the spread here is less than what it is for P2. So you're getting this. And so this here is epsilon. So this gives you the partition.

And so there's two things you have to be careful with, that for one function, you get one partition. For the other function, you get another partition. So you're looking at the subpartition that has all of the dividing points of both of them. And the other things you have to be careful is about-- and that was embedded here, but it was actually more clearly on one of the other boards, is that when you're looking at two functions, and you're looking at a little interval, and you're taking the sup of the sum, then that's bounded by the sum of the sup. But it's not equal to the sum, and likewise for the inf. Yeah?

AUDIENCE: [INAUDIBLE] for a partition because you always have to be [INAUDIBLE]?

PROFESSOR: I mean, so there's not. But the reason it's not is that once you have this general framework, you wouldn't really get back to the partition, actually. So if it was-- I mean, once you have this solid , foundations then you just write. You just write the integrals, right? Yeah.

And so, just on that note, this dx, this is-- so this here-- this here is-- this here is like-- it's referring-- this is like a giant S. So it's referring to summation. This here is, of course, a d. And delta is the Greek letter for d. And so this here, you should think about this here as referring to some sort of limit of this delta xi.

So that was-- let me just talk a little bit about the other two rules. So one of the other ones is-- one of the other ones is that if you take two functions, f and g again-- so number 3 here was that if you have f and g, they are both integrable on this interval here. And f here is less or equal to g. Then it implies that the integral here is less or equal to the integral of g.

I won't prove this. It's sort of, in a way-- maybe I'll just say words, that if you take a partition, then, of course, you're looking at some xi minus 1 and xi, some interval like this. And so, if you're looking at the function f, and you're looking at the sup of the function f on this little interval, well then, obviously, this sup here is less than the sup of g, just because you have this inequality everywhere.

And so this means that the-- and likewise for the inf. And so you see that it's this inequality. You have to, of course, make it nice, into a mathematical proof. But this inequality will seem kind of obvious. Let me instead try and prove that if you split the integral up into pieces, then you can just sum the pieces. So this is the following.

So again, in the notes, which I haven't-- I forgot to post, but I will post-- I guess these are posted because this is from-- these are things we didn't quite do yet last time. But the proofs of all of these four here, also the one in particular-- like the one, this one here, I didn't do now, they are all-- the proofs are all in the notes.

So number 4 here is that if you take a function, and it's integrable on some interval like this, it's bounded-- always bounded-- and you're looking at the integral here of this function, and c here is-- c is somewhere between. So c is lying between a and b.

Well, then, actually-- then, actually, let me just full write down this. Then the function here-- if you're think about restricting the function to the smaller interval from a to c, it's also integrable there. And likewise, if you restrict it to the integral from c to b, then it's also integrable here. So it's integrable on both these subintervals.

And you have that the integral here of f dx is equal to the sum of the integrals, like that. And so now, why is this the case? So this is just because, if you take-- so let's see. Right. So now, let's try to prove that.

So right. So I want to look at the-- right. So let me first prove that if it is integrable here, it's also integrable on these two subintervals. And then once we know-- in that proof, we will also see what that sum is equal to.

So I again use this lemma from before, that all I need is to-- for a given epsilon, I need to find a partition so that the spread between the upper and the lower sum for each of these two is bounded by epsilon. So given epsilon, since f here is integrable on the interval from a to b, we know that there exists a partition P such that the upper sum minus the lower sum is less than epsilon.

Then I look at a refinement. So here, I have a. Here, I have b. And here, I have c. And then I have some partition. But the partition may not have c as a dividing point. So I'm looking at-- so let's let P star be the refinement, that in addition to all the dividing points of P, also have c as a dividing point.

So it has one more dividing point. I mean, of course, it may be that c was already there. Then you don't-- then this partition is fine. And so now-- of course, now the thing is that once you have that, well, first of all, you have-- this is refinement. So you still have-- so you may as well just think-- so you have this inequality.

So you have that U of f, comma, P star is minus L of f, P star, is also strictly less than epsilon, right? That's because it's refinement of P. And so now you're looking at the Ps, so this thing here. But this thing here-- this thing here is just equal to-- this here is simply equal to U f of P star intersected with a, c, plus U P, U P star, intersected with the interval from c to b.

By this here, when I'm taking the partition, and I intersect it with the interval, I'm just thinking about the part of the partition that is dividing this interval up. So you have this. And likewise, this thing here is the sum. This one here is the sum of the partition, the lower sum-- the sum of the two lower sums, which is this, a to c, plus L f, P star of c to b, like that.

And so this, of course, means that-- so now you can split this up. And you can divide by-- so you have-- this is equal to this sum. This is equal to that sum. So you can just rearrange it so that this upper sum-- have. so this means that this whole thing here is equal to U f, P star, intersected with a, comma, c, minus-- and then I'll take this one here next-- so L f, P star, intersected with a, c.

And then I take plus this one here. And then I take the last one, which is this one here. And it, again, comes with a minus sign, so minus L f, P star, intersected with c, v, like that. OK?

But each of these things here-- this here is non-negative. This here is non-negative. And you see, the sum of these two numbers is bounded by epsilon. So this means that each of them must be at most epsilon, right? Yeah?

AUDIENCE: Would you able to explain again what you did to break up P star into the intersection of a to c and b to c?

PROFESSOR: OK. So I have-- I have here the integral. I have here the integral from a to b. Here, I have c. And the star partition is the one that has all of the dividing points of the original partition, P, and then also c.

So the partition looks like this. Maybe this here would be the partition. And now I'm thinking about-- so the sum here in the-- the sum from the partition, it would look like this. And I'm summing over all i, right? But I can look at where I'm summing over just those intervals, up to here. I can look at where I'm just summing over this interval. And then I can look at where I'm summing over those intervals.

And so that's what I mean. So this here is now where you're summing over all the i. But you can also just summing over the i's that are less or equal to c and the it's that are in this range, right? OK, so that's-- OK, so this here proves-- so what this proves is that if you take this function, and it's integrable on this interval, and you now restrict it to either of those, they are also integrable. That's what it actually proves.

But it's easy to see that it also-- with a little bit-- maybe I'll leave this for you-- but that through the proof, we also get that this sum here will have to be equal to the sum of these two. But really, what I, technically speaking, proved was just this part here. But again, all of the details for this are in the notes.

Now, the next thing I want to-- OK, so yeah. So I will-- let's talk about one other-- one corollary of this. So it's the corner, really, not of this fourth property, but on the third property. So the third property was this with-- so the third property was this property here, over here. So I want to talk about a corollary of this one here.

So now I have a function. I'm assuming both the function and the function that is the absolute value. So the function that's absolute value means that at each point, instead of looking at f of x, I'm looking at the absolute value of f of x. This is the function. So I'm assuming that both of these functions are Riemann integrable on this interval from a to b.

And then the claim is that the integral here of f is less or equal-- dx-- is less or equal to the integral from a. This is a very-- this is very simple inequality, but extremely useful. So this is the claim. So I'm already assuming that things are-- everything is Riemann integrable. So I don't need to prove anything like that. I just need to establish this inequality. And so, to do that, I'm just looking at f, where f is less than or equal to the absolute value.

If f is positive, it's equal to the absolute value. If it's negative, this here is non-negative anyway. So this is clear. This is clearly for all x. But it's also clear that if I take minus f, you see, this is also less than this.

But now, taken together-- so those two things-- so now, the first one-- so let's say star here and double star here. So star here plus property 3 from over there, this just gives us that the integral of f from a to b dx is less or equal to the integral of the absolute value. This is from star and 3.

But if we take double star here plus 3, well, then we have that-- so this here. So we have that the integral of minus f dx is less or equal to the integral like this. But this one here is-- you can just think about this here as minus 1. This is like minus 1 times this function. So minus 1 just comes out. That was like number 1.

So this here is like minus 1 times the integral from a to b of f dx. But this is, of course, the same as minus the integral from a to b f dx. So you see that-- so this here-- so the second here implies that minus the integral from a to b f dx is less or equal to this thing here. And so now you have this thing here and that thing here. And so, together, it just means that the absolute value of this is bounded by this. And so that's really this corner.

The next thing I want to talk about-- so this is how to evaluate integrals. And so there, you have the fundamental theorem of calculus. And so we'll give two versions of that. So how to evaluate integrals. And so this is the fundamental theorem of calculus.

And now, what does the fundamental theorem of calculus-- so let's give version 1. So version 1-- version 1 is that if you take a-- if you take a function, f, and f here is continuous, and you're looking at the integral-- you're looking at-- so you have a function. I don't really think about it as two versions.

But the version-- but I divided it into two versions here. So the function-- if f is continuous, say, on the interval from-- on this interval here, and you define a function, f of x, which is the integral from a to x of f of s ds, you see, for each fixed x in the interval here, well, if you're thinking about the function, little f, little f was continuous on the interval. So it's continuous on-- so f here is continuous on the interval from a to x.

So when you restrict it, it's still continuous. So this means that it's integrable. And so this here gives you a number. And so, for each x, you can think about this number here. So this gives you a function, capital F. And the claim is that this function here-- so the claim is that the function capital F, defined this way, is differentiable. So capital F is differentiable.

And the derivative of this at x is little f of x. And maybe I'll just-- yeah. So that's the claim. OK. Let's try to prove that. So let's look at-- let's try to prove-- so we want to prove-- so we want to prove that this function-- so we want to prove-- so we want to show that f prime exists at, say, at x0.

If you want to prove that it exists, then you need to look at the difference quotient. So we'll need to look at this difference quotient. And we need to prove that this thing here has a limit as x goes to x0. So x0 is fixed. And you want to prove that it has a limit, right?

So in order to estimate this, let me just-- let me not divide first. Let me just look at what this difference here is. And then, afterwards, I'm going to divide by x minus x0. So if I'm looking at this, let's, for the sake of argument, assume that x is bigger than x0. Of course, I need to also see when I'm approaching x0 from the other side. But the argument for that is similar.

So I just assume that x is bigger than x0. Well, this difference here-- well, the function-- so this function here, the function here-- so let me just write it over here so I have a little more space. So this difference here, where the function-- this here is the integral from a to x f of s ds. And this thing here is the integral from a to x0 f of s ds, right?

But this first integral here, I can use this fourth rule that I maybe don't have anymore. But the fourth rule is saying that you can split it up. An integral, you can split it up. So here, if you have a, then you have x0, and then you have x, if you want to integrate little f over this, it's the same as integrating over this plus the integral over that.

So I do that. So this one here, I break into two pieces. I integrate from a to x0 f of s ds. And then I take the second piece, which is from x0 to x, and then minus the integral from a to x0 of f of s ds. OK? But now you see that this one here cancels out with that, right? And so this difference is just the integral from x0 to x f of s ds.

So what we have is we have that this difference here, f of x minus f of x0-- we know that this difference here is equal to this integral here. Now, let me use this to bound this thing here from above and bound it from below.

You see, this integral here, this function here-- so this function here, if you're looking at this integral here of f of s ds, well, this function here is smaller than the max-- the function is continuous, so the max and sup is the same-- the max, or the sup, or whatever you want, on that interval. So this thing here is less than the integral of, if you will, x0 to x of the sup of f on this interval here.

But this is just a constant. And so this here is just-- this is just like the sup of the function on this little interval times-- so this here's a constant. So this just pops out. And then it's just the integral of 1 on this. But that's just the length of the interval. So this is this here times the length of the interval. And the length of the interval is this.

So this thing here is less than that. Likewise, this function here on this little interval is bigger or equal to the inf. So this is bigger than if you were just integrating the inf of the function on this same interval. But the inf-- this is just a constant. This pops out. So this thing here is just the inf of the function on this interval here times the length of the interval, like that.

So now I can-- and remember that this thing here was equal to the difference. So now I can just record this. So what I have is now-- what I'm interested in is in the difference in capital F. So what I'm interested in is-- of course, I really want to look at the difference quotient. But I can do this in just a second. So I'm interested in the difference of the function at those two points. I know that that difference is equal to that integral here.

And that integral is bounded from above by this and below by that. So I just write-- I can now just write that this difference here-- maybe I should write it so I have a little more space. Somehow, F of x minus F of x0, the difference is bounded from above by this thing here, the sup of f, on this little interval, times x minus x0. And it's bounded from below by the inf on that same interval times the length of the interval.

So now we can go to the difference quotient because we just divide by x minus x0 on both sides. x is bigger than x0. So you have this thing here. So you have this difference quotient here. And that's what we want to show has a limit. And now we have that this is less than sup of f on this little interval here. And it's bigger or equal to the inf of f on that same little interval.

But you see that as x here-- but now you have-- now you have that as x goes to x0-- so now, as x goes to x0, the values-- and not just the value at x, but actually, in this whole interval here-- the values goes to what the values is at x0.

So you see that this here, the right-hand side here, goes to f of x-- little f of x0. And the left-hand side goes to little f of x0. So this means that this thing is squeezed between these two numbers. That goes to the same number. So this difference quotient also goes-- so this difference quotient here goes to f of x0, because it's squeezed between those two numbers that converge to the same number-- namely, to f of x-- little f of x0, right?

So this here proves that it's-- but technically speaking, I only did it where I'm assuming that x was bigger than x0. If you let-- so you also need to check when x is strictly less than x0, and it's converging to x0 from below. But with trivial modification, the same proof works. And that's done out in the notes. So I will not go through that.

Now let's talk about-- so this is the second version of the fundamental theorem of calculus. And so-- again, you wouldn't really eventually think about it, probably, as two different versions. But they're slightly-- it's two things about the same stuff.

But the second version here-- So I divided it into these two versions. So the second version-- so again, fundamental theorem of calculus, version 2. And so this is the following-- that you take a function, F, that is defined-- and F is defined on some interval from a to b into R. And F is differentiable. So this here exists. It's differentiable at each point.

And f here, this little f, is Riemann integrable. And then the claim is that F of b minus capital F at a is the integral from a to b of little f of x dx. So that's the claim.

I wonder if I should just skip that. I did the proof for this out in the notes also. So maybe I'll just skip that. And then-- OK, so now-- but let me look at an example where I'm applying it.

And so I have-- I'm looking at-- so suppose I want to compute x squared from 0 to 1 dx. Suppose I want to compute that. And so, in this case, I could look at-- so if I want to apply that version, I could think about-- suppose I'm looking at F of x equal to x to the power 3 over 3. Suppose I'm looking at this.

Well, if I'm looking at this function, I know that capital F of x is equal to the derivative of this. It's just x squared. Right. So this, say-- and x squared is now-- of course, it's continuous. So it's integrable. So this means that by the second-- by this second version of the fundamental theorem of calculus, then we have that F of 1 minus F of 0 is equal to the integral of the derivative, because the derivative is integrable, so 0 to 1, of x squared dx.

But this here is now just evaluating this function here. So this here is 1/3 minus 0. So this original integral is 1/3. OK. So the next thing-- so the next thing I want to talk about is improper integrals. So this here concludes-- this here was stuff I actually had from the previous lecture. That was why I skipped a couple of things. But yeah, but the details for the things I skipped is in the notes. And the things I skipped is very similar to things I went through here, anyway.

So now, let me look at improper integrals. And there is a couple of versions of this. So we will look at two versions of this. One is where the integral-- where the integral is-- so we will look at improper integrals. So I'm going to look at two versions, so two versions.

Version 1 is where the integral is unbounded. The integral is unbounded. And the second version is where the function is unbounded.

Now, in the first version, then you take-- you're looking at a function, f. And this function is integrable. So f here is integrable from a to b for all-- so a here is fixed. But it's integrable for all b bigger than a. So a is fixed. So a here, fixed. And it's integrable like that.

Then what we want to-- what we are interested in-- so again, if you're thinking about the integral as a-- think about the integral as the weighted area. And weight just means that if the function is negative in a stretch, then the area there will count with a minus sign.

So this here is-- so what we're interested in is we were interested in making sense of the area, the weighted area below the graph, where we go all the way out to infinity. And so this is-- does this make sense?

So this is, in this case, the improper integral. It's improper because it's not a bounded interval. And so yes, if-- yes, if this limit here of a to b, If this limit here of f of x dx, if this limit here as b goes to infinity exists, then we say that the improper integral here from a to infinity of f of x-- we say that this here exists and is equal to this limit.

So this is one version of the improper integrals. And the other version is-- the other version of the improper integral is the following. So the second version-- so this is where f here is unbounded. In a minute, I'll look at examples of both things. So the function is unbounded. So it's defined. But f here is supposed to be integrable on, say, a to b.

Right, so a is supposed to be integrable from-- it's supposed to be integrable from c to b for all c bigger than-- so c here, where c is bigger than a and less than b. So think here about-- here, you have a. Here, you have b. So a and b, in this case, are fixed.

But the function is not bounded. It's not maybe even defined at a, right? It's not really defined at a. But it is integrable on-- if you go anywhere strictly to the right of a, then it is integrable on this interval here.

And so if this limit-- so if this limit here as c goes to a, the integral from c to b of f of x dx-- if this limit here exists, then we say that this improper integral from c to b-- sorry, a to b, of f of x dx-- this is an improper integral because the function is not bounded on this whole interval. It's not even defined at a. If this here-- we say that this thing here is equal to this limit as c goes to a of c, b, f of x dx.

And so the improper integral actually often appear in things. So let me look at examples of this. So I will look at example of this. And I will look-- so again, there's these two versions, where the interval is unbounded, or the function is unbounded. Of course, there's all kinds of variation of this. Here, I looked at where the function was unbounded to the left. But you could do the same to the right.

It could be this interval here, this interval in the-- where the interval was unbounded. I'm talking-- I was when I explained what it means, I did it as if the interval was unbounded to the right. But of course, you could also look at unbounded to the left, or you could even look at where it was unbounded both to the right and to the left.

But let me look at some examples. And so the first example-- again, the first two examples would be where the interval is unbounded. So this first example here is where the interval is unbounded. And we're looking at the interval is 1 to infinity. And we're looking at the function, f of x equal to 1 over x squared.

And I'm interested in-- and the question is, does the improper integral, this here, of 1 over x squared dx, does this exist? That's the question. And so we use the fundamental theorem of calculus to determine that. We observe that if we're looking at f of x, which is 1 over x, then the derivative here, of this function here, this is 1 over x squared.

So this is what we observe. So therefore, it means that if you're integrating from 1 to b of 1 over x squared dx, this thing here is the same as f of b minus f of 1. And so, using this, and we know what f is, you have that this is minus-- minus, sorry, 1 over b and-- no, sorry. Yeah, it's minus 1 over b. This is what you get from this. And here, you're getting 1, plus 1, like this, right?

And so you see that this thing here, as b goes to infinity, this has converged to 0. So this has converged to 1. So indeed, this improper? Yes is the answer. And what it's equal to, because it converts to 1, it's equal to 1.

So again, if you're thinking about it, it's a little bit like what we did for series that you're summing infinitely many numbers. So here, you're looking at something. And you're looking at the area. But the area actually is finite, even though it continues on forever.

So that's one version of it. Suppose that instead of 1 over x squared-- so 1 over the property that the function 1 over x squared has is that it goes really quite fast to 0. As x goes to infinity, 1 over x squared goes really pretty fast to 0.

Suppose I look at a function that does go to 0. But if it doesn't go to 0, if the function doesn't go to 0, if we kind of-- let's say that we're staying above a certain line. Well, then the area below the graph would, of course, be larger than this thing. And that would be infinite. So it's not going to converge. So it will have to go to 0 in some way.

So example 2, I'm looking at another function. Still goes to 0. Again, the interval is the same from 1 to infinity. And the function is now-- this time, it's 1 over x. It still goes to 0. So maybe it has a chance. Let's see. If you now take the integral from 1 to x of-- 1 to b, sorry, of 1 over x dx, well, again, you use the fundamental theorem of calculus.

If you're looking at the function, f of x equal to log of x, the derivative of log is 1 over x. So this thing here, by the fundamental theorem of calculus, this second version of fundamental theorem of calculus is f of b minus f of 1. And that is log of b minus log of 1. Log of 1 is 0. So this is just log of b.

But as b goes to infinity, this here actually goes-- it's not bounded. It tends to wander off to infinity. So this improper integral here, this one here, does not exist. It's like the area below this function here would actually be infinite, right? It does not go to 0 fast enough.

So that's examples of where the interval is unbounded. Suppose that we look at examples where the state of the interval is being unbounded. The function is unbounded. So let's look at some examples of that. Suppose you take-- so now, suppose you take the interval from-- so the interval is bounded. But the function is not unbounded.

Sorry, the function-- the interval is bounded. But the function is unbounded. So I'm looking at, first, f of x equal to 1 over square root of x. As x goes to 0, square root of x goes to 0. So this means that 1 over it goes to infinity. So this function is not bounded.

But if I'm now looking at-- so I want to look at the improper integral, where I'm taking this function. The function is blowing up as I'm going close to 0, blowing up like that. And I want to see, does this here-- does this improper integral here-- does this exist? That's the question.

And so, in order to figure that out, I use the fundamental theorem of calculus again. So if I take the function-- if I take the function, f of x is equal to 2 times square root of x, and I now take the derivative of this, then I get 1 over square root of x.

And so this means that-- so this means that if you now look at-- so this means that if you're now looking at the integral here of-- this integral from a, say, to 1 of 1 over square root of x-- a is strictly positive-- dx. This thing here is capital F of 1 minus capital F of a.

And so capital F was this function here. So this is 2, right? 2-- square root of 1 plus 1, so that's 2-- minus 2 square root of a. But now, as a goes to 0, this here converts to 2. So this improper integral exists. So yes. This one here exists and is equal to 2.

Let's look at one last example of this. And so this is another example where the function is-- so again, in this sort of example, the function was unbounded. It's blowing up. It's going to infinity. But the area below the graph-- that's where we calculated-- is still actually finite.

But what if the function is blowing up faster than this? So this is an example like this, that if you take-- it's, again, the interval from 0 to 1. And the function is now-- now it's going to infinite faster as x goes to 0. So this is the function.

And so now we want-- what we're again asking is, does the improper integral-- so does this improper integral here exist? And let's see if it exists or not. So we use the fundamental theorem of calculus again. We observe that the function, F of x equal to log of x, that the derivative of this function is just 1 over x. So this means that if we integrate from a-- a is now, again, strictly positive, strictly positive.

So if we're looking at this integral here, this here is-- because of the second version of the fundamental theorem of calculus, we have that this here is F in 1 minus F in a. And this thing is now log of 1-- that's 0-- minus log of a. Log of a is-- right, so log of a-- so this is like that.

But now, remember that the log function here-- the log function looks like this. This is the log function. So you see that as you go towards 0 here, this number here goes to minus infinity. So this here goes to infinity. So it doesn't converge. So the answer is no.

And so the very last thing I want to talk about is, I'll just give you a definition. So this is arc length. So suppose you take a curve in the plane. So suppose that I have a curve in the plane.

So gamma of t. This is a curve. So it has-- in the plane. So it means that it has two coordinates called gamma of s. I think that's what I did in the notes. So this here is the curve. And I'm assuming that f and g are differentiable. And the derivative here are continuous.

So suppose I have such a curve. And s here is running in an interval from a to b. Then the length of the curve is, by definition, the integral from a to b of square root of f prime of s squared plus g prime of s squared ds. So this is the length of the curve. Maybe I'll talk a little bit more about that next time.

OK. So I'll stop here. And I'll post the notes. Any questions? OK, great.