[SQUEAKING] [RUSTLING] [CLICKING]

TOBIAS COLDING: So now let's talk about, first, the different concepts that we have talked about in class. And then in the end, I'll get to just some example of some problems that you might face on the midterm.

So the first is this, that if you take a subset of R in to say that A is bounded from above, it just means that there exists M such that for all agent in A, a here is less than or equal to M. So this is what it means to be bounded from above from a subset.

And now the least upper bound-- and this is a number. So M here is at least-- so M is an upper bound.

If M is an upper bound and any other upper bound-- let's call it M bar-- is equal to M. So this is what it means to be least upper bound.

And so the least upper bound, this is what we denote by sup of A. We'll say that there is the [INAUDIBLE] upper bound and similarly for lower bound and inf.

And the real numbers have this property that if you set this bounded from above, then there is a least upper bounds, and likewise for a set that is probably from below.

OK, so that was sup. And we'll get to this with lim sup and lim f. I will just review that a little bit later also. But that's another form of chain. A sequence-- so another key concept in this part of class is sequence.

And we'll just begin with a sequence of real numbers. So this is a map from the positive integers into r. And again, we denote-- so the image-- so if you're looking at the image of some n, we typically denote this by a n or by a n or xn or something like that, where we use this subscript.

But it's just the image under this map. So that's a sequence. So for each positive integer, you get a real number. Then for a sequence, one of the first things you were interested in is convergence.

So if this here is a sequence, that xn converges to some x here, this just means that for all epsilon greater than 0, there exist-- I will use this symbol. I haven't used it until now, but it's just so hard not to use it. This here means there exists-- it's not on the midterm.

But it's a pretty standard symbol. So there exists a capital N such that if little n is equal to capital N, then xn minus x is smaller than epsilon. So this is what it means for a sequence to converge to x.

Sequence xn is equal or greater to x. And we write it like this. Or you can write xn and then the limit here as n goes to infinity is equal to x. Both of these are standard use.

And then also, if a sequence is not converged, then we say it is divergent. So the sequence is either convergent or it's divergent.

And then you have this property, useful property, so a lemma If xn is convergent, then the set-- this here as a set-- this here is a bounded set. So we'll talk about that.

And then another thing about sequences is that the algebraic properties of sequences-- and what are the algebraic properties? They are that if you take two sequences a n, b n-- and let's say that a n converge to a and b n and converts to b.

And c let's is just a constant. And often, when you combine what I did, those one and two before-- you combine. You're looking at a and plus c times b. And this here converge to a plus c times b.

This is the first. And when I state it like this, it actually combine two of the one properties that we talked about earlier. The second one is that a n times bn will converge to n times b.

Third property is that if a n, if this here are not equal to 0 and the limit here is not equal to 0, then 1 over a n converge to 1/a. And so, of course, this last one, this one implies in particular that also bn over an inverse b/a. This is just because this property here combined with the property 2. That would give you this.

OK. So that's the algebraic properties of limits. And then you have the concept of a subsequence, so a subsequence. So we imagine that we have a sequence. And often, we don't go all the way back like this. But a sequence, again, is a map from the positive integer into R. And we denote the image of n typically like xn.

Then a subsequence of this sequence-- and this is typically denoted something like this, where you have a further subscript. And this is important, again, that this is like equal to f of g of k. Whereas g here-- and this is a really important property-- that g is a map from the positive integers into the positive integers. And then really important is that g here is strictly increasing.

And so strictly increasing guarantee you that these numbers here is taken in the same order as in the original sequence. But you may skip a bunch. You take one, and then you may skip a bunch. And then you take another one. And then you may skip more. But it's in the same order. And that's what strictly increasing is guaranteeing.

And then you have this property that's-- so you have this theorem that a sequence xn is convergent if and only if all subsequences are convergent

And one of the two directions for this here was trivial. Because if you take a sequence, then it is-- if you're thinking about this map here, g, as being the identity-- so that would give you a subsequence. But that really is the original sequence. So the original sequence is a subsequence of itself. So if all subsequence are convergent, well then, in particular, the sequence itself is convergent. So one of those two iterations was trivial.

OK. Now then we had these two theorems. So the thing in mathematics is that things build on each other. And hopefully-- each step is fairly trivial, but you build step by step. And then you end up with something highly nontrivial. So this here was pretty trivial.

But then we used it later on to prove something nontrivial. So this was this monotone convergence theorem. And there was two versions of this. You had a sequence xn. And xn was increasing.

So that means that the next element is bigger or equal to the previous element. If this here was the case, then xn is convergent if and only if xn is bounded, if the sequence is bounded.

And the sup of the xn, this here, this is the limit. This here is the limit.

And the second version of this is, if you have a monotone-- so instead of being increasing, it's now decreasing. So the next element is smaller or equal to the previous element. And then similarly, the sequence xn is convergent if and only if xn is bounded from below.

And in this case, and if it is bounded from below, then the limit here of xn is inf. OK. So that was the monotone convergence. And it has these two versions, the increasing version and the decreasing version.

Then another important concept was that of a Cauchy sequence. And the reason why Cauchy sequence is useful, even for the real numbers-- later on, we generalize these concepts.

It was obvious demonstration of the concept of convergence, but also of there being a Cauchy sequence to metric spaces. But even for the real numbers, Cauchy sequence, you might be interested in whether a sequence is convergent. But sometimes, it's much easier to prove that it's a Cauchy sequence than to prove that it actually is convergent.

And so if you don't really care about what the limit is, then it's often easier to prove that it's a Cauchy sequence. So Cauchy sequence, so you have a sequence-- again, we're still in the real numbers. And this is a Cauchy sequence.

It just means that for all-- and now I will use also this here. This symbol just means "for all." And that's also a standard symbol. It's an inverted A.

For all epsilon greater than 0, there exist-- again, this here, inverted E, means there exists a capital N such that if little n and little m are bigger than capital N, then those two elements are bunched epsilon close together. [INAUDIBLE] This is what it means.

For a sequence to be Cauchy sequence, then the tail of the sequence lies very, very close together. Then let me skip-- in my notes, I have something about contracting. But actually, there won't be anything about contracting that. So I'll skip that.

Let me next talk about the Cauchy convergence theorem and the Bolzano-Weierstrass theorem. So Cauchy-- also, let me just remind you that, again, on the midterm-- so there are these five main problems. Then there's the six bonus problems.

But make sure that you-- it doesn't matter which one you do. So you might as well start with the ones that seems easiest to you. And you can see here where it says exactly how many points each of them give. So if you don't have enough time to finish all of it, you can make sure to try to optimize.

OK. So what is the Cauchy convergence theorem? So again, we are in R. And we have n. So the Cauchy convergence theorem [INAUDIBLE]. Let's say that if xn is a Cauchy sequence, then xn is convergent.

That's a Cauchy convergence. Then for Bolzano-Weierstrass theorem-- and this Cauchy convergence theorem, we talked about that. So the concepts, when we in a minute review metric spaces, then convergence sequence, that makes sense in the metric space. Cauchy sequence makes sense in the metric space.

And the statement of the theorem makes sense in a metric space. But it's not always that it holds. It holds in R, but it doesn't hold in a general metric space. The Bolzano-Weierstrass theorem statement of that also makes sense in a general metric space, but it doesn't hold in a general metric space.

So again, these are things about the real numbers. They extend to more than a real number. But for other spaces, you have to prove it individually. So Bolzano-Weierstrass theorem says that if xn is a bounded sequence, then xn has a convergent subsequence.

OK. So that's the Bolzano-Weierstrass theorem. And the next thing I want to review is what it means for a function to be continuous. So continuity is another important property. It's closely tied together with this whole sequence. And we'll get to that in just a minute.

So if you have a function-- let's say on R, but it could be defined on a subset of R-- so if you have a function like this, then we say that f here is continuous at x0 if for all-- it's continuous at x0 if for all epsilon greater than 0, there exists a delta greater than 0 such that f x minus x0 is smaller than delta.

Then the image is smaller than epsilon So that's what it means for a function to be continuous. And there was this theorem or lemma that tied things together.

So lemma-- if f is continuous at x0 and xn converges to x0, then the images of the sequence-- so that's another sequence-- converges to f of x0.

So we proved this even in a metric space. But normally those two are not. OK. And then we had the algebraic properties of functions. And they were very similar to this. So let me just review them, so algebraic properties of continuous function.

So continuous functions-- so imagine that you have f and g are continuous functions. So everything is defined on R. c here is a constant.

Well, then property one that corresponds to that property one is that this function plus c times this function here, this here is also continuous. And then two is that if this function here is never equal to 0, then 1 over this function here, this is also continuous.

And sorry, that was not true. Sorry. I shouldn't have made that mistake. I should make that the third property, sorry-- that f times g is also continuous. And then the third property would be that if f is not equal to 0-- that would correspond to that-- then this function here is also continuous.

And again, I divided it up when I did it. I divided it up into four steps to separate this out into two steps OK. And so from this one, we got that-- just because then we proved that the constant function was continuous. And the linear function, the function f of x equal to x is continuous.

Once we have that, then we just use one and two to prove that all polynomials is continuous. And then if we bring in three, then we have this with all rational functions are continuous except, of course, where the denominator vanishes. OK. So that was how you build continuous functions from noncontinuous functions.

And then we proved the extreme value and the intermediate value theorem. We had these two theorem. And they were proved-- so one way to prove it that we did last time was to use this lemma that tie together continuity and sequences.

And so we had the extreme value theorem. And this just say that if you have a function-- but now it's crucial that this here is a bounded interval and the endpoints are included in this. So if you have a function like this, and this function is continuous, then the sup and the inf over the set here is achieved or, if you will, the max of f and min of f achieved.

In particular, the function is bounded. So I'll just state it like this with a brief, like that. So that was the extreme value theorem. The intermediate value theorem is saying that if you have a function-- imagine that starting off here and ending there. So this is the function.

The intermediate value theorem say that everything in between these two is also the image of something. So let's say that f of a is smaller than f of b. And y here is between f and a and f of b. Then there exist x between a and b so that f of x is equal to 1. Again, the functions we're talking about are continuous, and they are defined on this interval.

OK. So that was those two theorems. And they used some property about that-- they used two things. They used the Bolzano-Weierstrass theorem. And then they used how sequences are continuous functions, both tied together. The Bolzano-Weierstrass theorem is using something about that the real numbers are ordered and like this with sup and inf. You have the least upper bound property and the greatest lower bound property.

OK. And another thing-- so we talked about-- so some of the highlights, in terms of concepts, are sequences and continuity of functions and then various properties that follow from this. Then another thing is series. So that's another key concept.

So series-- so series is when you're starting with a sequence. So series-- so you're starting with-- so this here is a sequence.

And now you form another sequence, Sn. But here, you're summing. And it could start wherever you want to start it. Let's say it start from 0. You're summing the first-- in this case, it would be, because you're starting at 0, you're summing the first n plus 1.

So ai-- and this here is a0 plus a n. We just write n plus 1 in this case. You don't have to start at 0. You start at 1, it would be the first n numbers. So you're looking at-- so a series is really kind of the same thing. But you think about it a little bit different. But it's really the same thing as a sequence. But you're starting with a sequence, and then you form a new sequence.

And then the key question here is-- the key question-- is the series convergent or not? So that's a key question.

And if it is convergent-- and the series is often written like this. And so this here will symbolize the series. And if the series is convergent, it will also symbolize the limit. So it has two meanings.

But even if it's not convergent, then-- well, if it's not convergent, you would just write down and you would think about this S series. Now there are some series that are particular important. And they can then be used to determine whether or not other series are convergent.

So let me just review them. And the most important of all is the geometric series. And the geometric series is the series c to the n, n equals to 0 to infinity. And c is just some fixed number, real number. And you're looking at this.

And this was the geometric series. And this is convergent if the absolute value of c is less than 1 and divergent otherwise.

And if it is convergent when c and absolute value is less than 1, then this sum here is equal to 1 times c. But remember two things. One is that when you actually want to evaluate the sum, it's crucial where you start. If I started at 1, then it's not any more equal to this. But you have to subtract the first element that was missing.

So for convergence, it doesn't matter where you start. But for the limits, it's key. That's one thing. And the other thing is that when you-- again, we talk about this, that when you're given a series, even before there's any question, when you have a series, the first thing you look at is, does this a n converge to 0 or not?

If a n does not go to 0, the series is not going to converge. There's no need to apply any fancy test. And for a series, the question is always whether it's convergent or not. So that's a geometric series. And then there are two other series that we talked about.

And let me just-- not only talk about, but we also determined whether or not they were convergent. Then we used them to determine whether other series were convergent. But again, the most important of all of them is the geometric series. That's the one that we use to determine whether power series in general are converging or not.

But there are two others that are worth mentioning. One is the harmonic series. The harmonic series is the series where you're summing 1/n. So this here is divergent. So this follows second.

And then there's a closely related-- let's call it 2 and 1/2-- that's called the alternating harmonic series. So the alternating, it was like the harmonic series, but the sine of it was alternating. And this here is actually convergent.

So there's 1, 2, 2 and 1/2, and then there's 3. And that's this series here. We look at this series here. And this here was convergent. The only one where we determine what the sum was was actually the geometric series.

OK. So that's some key series. Then when you have a series, there was one concept that we proved to be quite useful. And that was the concept of absolute convergent of a series.

And so you're looking at a series. And absolute convergence was that this sum here was convergent. So this here is absolutely convergent if this series, where the n replaced by the absolute value was convergence.

OK. So of course, for a series of non-negative numbers, if it's convergent, it's automatically absolutely convergent. And we talked about it a little bit, that one of the things about absolute convergence is that you could kind of reorder the way you're summing things, and it wouldn't matter for the limit. But if it's not absolute convergent, that really matters.

We thought about it. We didn't really prove anything there. But I think it was kind of clear, you could do this. One thing where we used it was that when we look at products of series, then the product of series was convergent if each of them was absolute convergent.

And there, you saw that we estimated some remainder term. And there, it was really kind of key that we had absolute convergence. So anyway, so absolute convergence is an important-- it's a stronger property than convergence. So absolute convergence implies convergence. Maybe I should write this there.

So we had the theorem, so absolute convergence. So this implies convergence, but it's stronger, generally stronger. OK. And then we had some tests. Then we had some tests.

And so we talked in particular about-- there was a few other tests that we talked about. But they're not actually going to be on the midterm. And some of them, we'll get back to. This one here won't be on the midterm. You'll have the alternating.

But we talked about how to prove the convergence. And when we talk about it, then we discovered that this thing here, that it was a stronger statement that if you have alternating and the a n were all non-negative and these guys was decreasing, decreasing down to 0, then actually, any alternating sequence like that would be convergent.

Anyway, but for the test, we really talk about was three tests. One of them had two parts. And then in a minute, we'll talk about how we apply them to power series [INAUDIBLE].

So we had some tests to determine whether or not a series-- OK. So what are the tests? So there are the comparison tests.

Let me just first write down the test. So the comparison test-- and there was two versions, version a and version b. Then there was the ratio test. And then there was the root test.

The ratio test and the root test, they are both offsprings of the geometric series. And both of these, you got-- let's just talk about this first. The ratio test was looking at this, the ratio of a n, [INAUDIBLE] divided by a n, a n plus 1 over a n.

So you are implicitly assuming that the a n does not vanish. I'm not sure. So you can form this ratio. And then as n goes to infinity, if this limit here is less than 1, then the ratio tests say that the series was convergent. If the ratio was bigger than 1, then this ordinary element, a n, would not even go to 0.

So in this case here-- maybe I'll write it over here. So the ratio test, you assume that the a is not if the a n equals 0. And you're looking at a n plus 1 over a n.

And if this limit here-- let's assume that the limit exists. Of course, it may not exist, right? But if the limit existed-- so if c here is less than 1-- it's always non-negative because these are non-negative numbers. If c is less than 1, then the series is convergent.

If this c here is strictly bigger than 1, we should find that the series is divergent. And if the c was equal to 1, then it's unclear-- and so unclear, which means that further tests are needed.

And so that's the ratio test. And the root test are similar. The root tests were-- so you're looking at the a n. You take the-- so this is the root test. And again, this is tested by comparing these to the geometric series.

So the root test-- so you the n-th root of this. And you're looking at the limit here. And we actually looked at the lim sup. I'm writing the lim sup. The reason why I want the lim sup is that I want to make sure-- later on, I want to apply more generally.

So if the lim sup here was less than 1-- lim sup is called a c. If c here is less than 1, then the series is convergent. Two, if the c is bigger than 1, series is divergent. This fails the most basic test, divergent.

And again, three, if c is equal to 1, then it's unclear, failed test [INAUDIBLE]. That was those two tests. The comparison tests, let's get back to that. The comparison test had two parts.

Suppose you had a n and bn. The first part is this statement, that if you have two series, one based on the a n, another series based on the bn, if you have this, if bn-- so this here, if this series here is convergent, then it implies this is convergent.

And this is sometimes used both exactly like this, and it's also sometimes used where you know that this one here is divergent. And so you compute this one [INAUDIBLE]. The other part of the comparison test was if you had the limit here, if this limit here exists and it was some f. then actually, it said that-- let's say L here is not equal to 0.

Then if one of them is convergent, the other one is also convergent. So if a n-- or bn, let's say-- bn is convergent, it implies that a n is convergent and vice versa. This actually prove that this here is convergent, if you [INAUDIBLE]. It's convergent.

So those were the three tests artists that we really went through. There was a few other tests, like, again, this alternating series test that you saw the proof, but it's not going to be on midterms. And we will return to this later. Yeah?

STUDENT: Are there [? patterns ?] of when we should apply which tests?

TOBIAS COLDING: Yeah. No. And I want to get to this in just a second. Let me just say a couple of words first. Let me jump to some of the examples of problems. And then I'll come back and I'll review a little bit more.

But let's just talk about power series-- and maybe a little interesting whether-- just now, when you were asking the question-- no. Let's just review the power series. So our power series was a series, but it was something that also defined a function.

So you have a power change. So you have a sequence a n. And then you're saying for each x-- sigma x is being fixed. You form a series. And this is the series that you form.

And so for each x, you can ask, is this series convergent or not? This here is called a power series. And the key point for power series is, again, of course, whether or not this is convergent. And this gave rise to the radius of convergence.

And so the radius of convergence was that we defined-- we looked at lim sup of the n-th root of a n, like that. And we looked at if this thing here actually defined this thing here.

And so we get this thing here. And then we define the radius of convergence. So the radius of convergence, this is r. And r is 1 over this [INAUDIBLE]. But there was an understanding in this.

The understanding is that this number here, this number potentially could be infinite. If it was infinite, it meant that this thing here was 0, if this one here is this infinity. If this here is 0, then this here would be infinity. So that's like a convention in this particular case. Usually, we don't want to divide by 0 [INAUDIBLE]. But this is a convention in this particular case.

And so using this radius of convergence, we saw that if you take a power series, if you have any power series like this, then you have [INAUDIBLE]. Any power cells like this, it's going to be-- if this is R And again, R could be 0, or it could be infinity. But if this is R-- sorry, I should write this minus sign.

Then for x in between, for x not including the endpoints but in between, the series is convergent. The power series is convergent. And outside here, this is divergent. Outside is divergent. Here is convergent.

And at those two points here, unclear. Other tests are needed. So this is always this picture. So this comes back to your question about which test-- of course, you can use any test you want. If you're looking at a series, you can use whatever test you want. But if it was a power series, you already know that there's some interval around 0 where it's convergent. And then there's a question about the endpoints.

OK. So now let's look at some examples of this. And suppose you look at-- and so it's usually not too hard to use either the root-- to use either-- this here was based on the root test. So you're using the root test or the ratio test for power series.

But you can take advantage of it. Sometimes it's just slightly easier. So let's look at two examples here. So suppose that you ask to find the radius of convergence for the power series.

OK. And then let's say that this is n to the fifth power xn. So suppose that this is the power series you're given. And you want to find the radius of convergence. You can either just plug it into this test. Or you can just say, OK, we'll just apply it. We know that it's some interval. And I just want to see, for which x, does this here converge?

So if I looked, in this case, you might find it slightly easier to do the root test. So if you're looking at the n plus 1 element in this series-- the n element, sorry. The n element would be this to the fifth power. Think about x as being fixed. And you're asking, for which x? And then for a given x, you want to determine whether this series here is convergent.

So if you're using the root test, you would look at this thing here. And you might find, in this case, slightly easier to use the root test. So if you're looking at this, this thing here is now n plus 1 to the fifth power times x to the n plus 1 divided by n to the fifth power times x to the n. That's that ratio.

And so, of course, you can write this in here as just n plus 1 multiplied by itself 5 times divided by n multiplied by itself 5 times. And then this thing here, you can write it as x to the n times x over x to the n. So you see, this thing here, you can then think about as n plus 1 over n to the fifth power and times x.

And I can write this in here in the usual way. If you want, you can write it as 1 plus 1/n. I'm dividing by n in both denominator. And denominator, you can write it like this, to the fifth power times x.

And so now the root test, it's asking whether-- the absolute value of this root test would be then asking if the limit here of xn plus 1 over-- oh, sorry a n plus 1 over a n, if this here has a limit.

And then it's asking you-- if that limit is less than 1, if it has a limit and the limit is less than 1, then the series is convergent. And it's divergent if it's bigger than 1. But in this case, this thing here, this ratio here is 1 plus 1 over n to the fifth power times x. This is a non-negative. So the absolute value is just going over here.

The algebraic property of the limit-- say that this one here goes to 0. And so raised to the fifth power still goes to 0. It's multiplying a bunch of sequences 5 times together that goes to 0-- to 1. So this here, this here goes to 1. And so even raised to the fifth power, still goes to 1. So this here goes to absolute value of x.

And so the ratio test would be that if x here-- so the ratio test gives you is that if this norm here is less than 1, then this series is convergent. If the norm here is bigger than 1, the ratio is divergent. And it's unclear if the norm is equal to 1.

So in other words, you see that the ratio test here gives you directly that the radius of convergence is 1. And, of course, you could apply the root test. The root test is great on this here. It's a little bit more-- I mean, you have to do a little bit more argument. It's not very hard. But actually, in this case, the ratio test would be less to argue than in the root test.

Let's do one more example of this. So maybe I'll just switch a little bit, and then I'll come back and review some more stuff. I'll just talk about a few of the problems that I added to the end of my review notes. Again, I'll post this with these problems. And obviously, they are-- it's not exactly the problem from the lecture. But it's certainly [INAUDIBLE].

OK. And also, another thing I want to say is that the homework is graded up to and including 4. So I wasn't actually aware of, over the weekend, it was supposed to be handed back within, latest, a week of when you handed it in. And I just discovered it over the weekend, that some of them were late. But it's fixed.

I'm not the grader. But there's two UAs that are grading and a grader. But it's all fixed now. So the first four, you can review those also for the [? lecture. ?] So the other power series I want to look at is-- so again, what is the radius of convergence?

The nice thing about power series is that power series is also a test about series. So it's like [INAUDIBLE]. And so what is the radius of convergence? And now the series is a slightly different series. It's this series here, so this series.

OK. And so now let me again-- I mean, you could apply the root test. The root test is the one that it always give you an answer. But you may have to work harder to find the n-th root.

And sometimes it's just easier to use the ratio test. So let's use the ratio test again. So a n here is n to the fifth power x to the n over n factorial. So if you're looking at a n plus 1 over a n-- maybe let me write over here.

So if I'm looking at a n plus 1 over a n-- so then I have the n plus 1 to the fifth power times x to the power of n plus 1 over n plus 1 factorial.

That's a n plus 1. And then I have to divide by this. But when I divide by a fraction, it's like multiplying by the inverse of this, so like this here-- n to the n power x to the n, like that.

And so, again, I can write this in here as-- so if I just do like I did before, I would have that n plus 1 over n to the fifth power. That's when I take this one here and that one together. Then when I take those two together, I can get rid of the n here. And I just get an x.

And then when I look at the ratio of these two, this one here, n plus 1 factorial, you can think about that as n plus 1 and then times n factorial. And so this n factorial cancel out with that, and you're left with an n plus 1. So this is what you get.

And this one here, just writing it as we did before, dividing this fraction by 1/n in both the nominator and denominator and getting this thing here-- to power 5 times x, n plus 1. And now you see that if-- and this here, I guess it should have been absolute value [INAUDIBLE]. So it's absolute value on this.

This one here converge to a 1. This one is just x, so this converge to x. And then this here goes to infinity. So the ratio here goes to 0. I'm using the algebraic property of limits. So this converges to 0. And so you see that it does have a limit. So the ratio test apply. Because that limit is 0 for all x, it's going to be converted for all x.

So the radius of convergence in this case-- and, of course, on the midterm, don't just say that the radius of convergence is infinite. Explain this in here. And the more detail, the better. Because even if there's some typos, you should definitely get-- I mean, I'm not grading the midterm either. But you definitely should get partial credit. Yeah?

STUDENT: So to get R, is it always like you divide by the lim sup? Is the thing on the bottom one the lim sup or just the lim?

TOBIAS COLDING: Yeah. So when you take just a complete general series, then the radius of convergence is, by definition, this. This is related to-- you asked this question a couple of times ago that-- because we started with the ratio test. We did that before we did the root test. And you were noting that in the ratio test, you're always comparing an element to the previous one.

And in general, the series can be convergent even though the next element is slightly larger than the previous. If that keeps happening, it's not convergent. But it could happen a few times, and then it goes really down. And that's fine. And so that's why the root test gives you this general result about that it always works. It doesn't assume that a n are not 0, et cetera.

For any particular-- and it really should be with lim sup here. And then it's defined. It's a number that's always defined. And again, the way it's stated here, the way the proof goes, it's not comparing two consecutive a n.

In any particular example, the root test may not be the easiest way to argue. Often, if you can, for a power series, almost always, I would say-- in a power series, it's almost always easier to use the ratio test. But it requires that the power series has some nice property for this to apply. Yeah?

STUDENT: So the reason you have that R as infinity for that example was because you know it's convergent for all x, not [INAUDIBLE]?

TOBIAS COLDING: So in this example here-- yeah. This guy is infinity, which means that this thing here would be 0. Yeah.

STUDENT: But you know it's very--

TOBIAS COLDING: But I didn't actually calculate. You see, I didn't calculate the [INAUDIBLE] in this example. But if I had calculated it, I would actually-- and it's not very hard to calculate, but it needs some small tricks. And yeah, we haven't talked about it. So that's why I didn't [INAUDIBLE].

Let me talk about a few other things. I didn't necessarily-- I don't think I can probably review all what I had in my review notes, but let me just talk about a few of the problems that I had. So let me just quickly mention the things I may or may not get to that I have in my review notes.

So I had also [INAUDIBLE] what is lim sup? What is limit final? So definitely review that also. We talked about the radius of convergence. And maybe this is all. And then we talk a little bit about the power series for the exponential functions. But again, all the stuff we did on the power series for the exponential functions-- and at this point, we have established this.

There won't be anything particular about-- I mean, the exponential function stuff could be on the midterm. But it's not like you would have to prove the properties of the exponential function, this property that it has, the property that has the exponential function x plus y is a part of-- that kind of stuff, It's not on the midterm. So it's not part of this review.

The other thing that I probably won't get to-- because in a minute, I'll just turn to these other problems-- is that metric spaces. Definitely review metric spaces. It was in the last lecture. So there's three properties for metric space. The distance should be non-negative. It should be symmetric. And the distance between two points should be 0 if and only if those points are the same.

And then the crucial property in the metric space that's the powerful property is the triangle inequality. And so there were a few examples. And review those examples. They are in my review notes. The most important example, I would say, is the [INAUDIBLE] metric is kind of crazy metric. That's not one of the important ones.

So I just did a review note, says just the most important ones. And then there's this. What does it mean to be convergent in a metric space? And what does it mean to be a Cauchy sequence? And so now let's get to a couple of the other problems that I suggested that you look at.

And so here are two other problems.

Again, I won't get to-- there's also problems about metric spaces, and you should definitely look at it. But I don't have the time [INAUDIBLE], even though it's relevant for the midterm. I just had to pick some. Again, all of the problems that I-- I have four problems here. All of these are from midterms.

So now suppose you take a sequence. So I have a sequence here, xn. And so often, a sequence is defined recursively, so recursively defined sequences.

So what does it mean? It means that, typically, the first one is defined to be whatever. And then the next one is defined from the previous one. And so the one I was suggesting looking at here is that-- you said this here, the first one.

And then actually, let me-- and then the next one-- let me just do this one. It's actually simpler than the one I have in my notes. Let me do that one. So the next one is defined to be xn over n.

So now the question would be, for what a does this have a limit, the sequence have a limit? And so you see that in this case-- we only have four minutes left. So I will save that for you.

But I just wanted you to review what it means for a sequence to be defined recursively, OK? And so you see that in this case, you have that this thing here, it is going down. It's actually going down. It even could go down by a fraction.

But now it's like, if you're looking at this thing here, if you're looking at xn. This here is definitely going to converge, whatever you start with. So I claim that you will always get xn here converges to 0. And you see that when you have something recursively, it often can pay to just write out the first few terms.

So you have x and a. x2 is equal to a. It's also a because it's the previous one. x3 is now a/2. x4 is then a/2 divided by 3. So this is a/6 and so on.

So if you continue like this, you will see that they get-- that it's like, a divided by some larger and larger number. But you have to, of course, write it nicely out as a proof. So this is proving that. But again, this here, if you just did what I did, that would not be enough to get full credit. So definitely write it out.

The other thing-- in the last minutes here, the other thing-- same here [INAUDIBLE]. Suppose you take a series like this. And suppose that a n is less than 1 over n squared. And then the question could be, is the series [INAUDIBLE]. If we knew this, but that's all you know is the series. Convergent or divergent? And then I say in the review here that either you should prove that for all such series it's convergent or, if that's not the case, you should give an example of a series with this property that is divergent.

Because obviously, if all the a n was 0, it would be convergent. But it still could be another series with a n. That property, that was divergent. But in this case, it is going to be convergent. And why is it convergent? Because this series here is convergent. We've already seen it. So this would be like the simple comparison test, like the first comparison test.

So these are just examples of things you should be reviewing. These things here, the two things to the end-- this one here-- this is super fast. So don't expect it to be quite as fast on the midterm. But it just gives you an idea of what you should be doing. OK.