[SQUEAKING]

[RUSTLING]

[CLICKING]

TOBIAS COLDING: OK. So today, we want to talk a little bit more about sequences and how sequences connect to a function being continuous. And then the second thing we will talk about is metric spaces.

So let me just remind you first that if you take a subset here, say, of R and you have a function A into R, then f is said to be continuous if for all x0 in A, the function is continuous at x0.

And the function is continuous at x0 if for all epsilon greater than 0, there exists delta greater than 0 such that if the two points-- so x0 and this other point, x, they deviate at most by delta.

Then their images deviate at most by epsilon. And in general, this delta here depends not only on x0 but also on epsilon. So delta would in general be expected to depend on both of these things. So we talked about this a number of times. And we will also generalize this later in the lecture.

So this is what it means to be continuous, that it needs to be continuous at all points. And this is what it means to be continuous at a point. Now, two things we talked about before is the extreme value theorem and the intermediate value theorem.

And I want to think about these two theorems in terms of sequences. And that's often a really useful way of thinking about it. So the extreme value theorem is that we have a function-- so the extreme value theorem, abbreviated like this, that's the theorem that says that if you take a function on some interval from a to b-- a and b are both included into R.

And this function is continuous. Then the claim is that then there exist-- so one way of formulating it is that then there exists xM-- capital M. This is something in that interval.

So there exists some points that we call x capital M-- and x little m also in the interval. And the first one has the property that for all x in the interval, you have that f of x is less than or equal to the value at xM.

So you have this. And you also have that f of x, that this here is bigger or equal to the value at this other point. So this means in particular that the maximum and the minimum is achieved. It also means that the function is actually bounded on that interval. So that's an extreme value theorem.

And we talked about that before, but I want to think about it in terms of sequences. And the other thing I want to think about in terms of sequences is the intermediate value theorem. So the intermediate value theorem say that all values-- let me just formulate it loosely.

All values between-- sorry. f here is as above. So you have, again, a function from the integral, like this, into R. And this is continuous. So you have this continuous function.

And then the intermediate value theorem-- so loosely formulated is, say that all values between f of a and f of b are achieved. So this means that if you take a function-- so you have a function. This here is the value at the endpoint.

Here, you have the value in the beginning. Then it say that everything in between these two-- these here-- is the image of something. That's what the intermediate value theorem say.

OK. And I want to prove these two using sequences. And so in order to do that, we need the following lemma, which is often-- so, again, this here is like the definition of being continuous. But often, when you're proving that something is continuous, you would instead use this lemma here.

So it's really an if and only if statement, but I will just prove the if statement. So if f is continuous-- so it's only the if statement I need in this case here. But in other cases, it's useful to have the if and only if statement.

So if f is continuous-- let me just say where the function is defined. So in this case, it's just defined on some subset of r.

If this here is continuous and xn is a sequence in a and xn converge to x and x is also in a, then the claim is that then the sequence here converge to this.

And remember that if this here is the case, then we write this here as x is the limit of xn as n goes to infinity. This statement here can be written as that f of x is the limit as n goes to infinity of this thing here.

And so if you combine these two things, if you fill in x is equal to that, then this here can also be written as that f of the limit of xn as n goes to infinity-- this is this by filling in this-- and then equal to the limit.

So in a way, if you want to think about it like it's something about-- that if it's continuous, you can exchange the order of taking limits. OK. Yeah. But that's a more formal way, in a way. But I was just thinking about it like this. OK. So we want to prove this lemma. And so then we want to use this lemma to prove these two theorems.

OK. So now let's prove this lemma. So we want to show that f of xn converge to f of x. So I'm first going to use that f.

So use first that the f is continuous at this point, x. So this is some point in a.

So this means that, therefore, given epsilon greater than 0, there exist delta greater than 0 such that. If x minus-- so any other element, y, if this thing here is less than delta, then their images deviate at most epsilon.

This is just using the definition that x here is now thought about as being some fixed point, a fixed point. And then y is like any other point. OK. So we have that. But then, second, we use that the sequence xn converges to x.

So this means that now thinking about this delta here that this epsilon is spitting out-- so this means that for delta, there exist capital N such that if little n is bigger than the capital N, then actually, xn minus x is smaller than delta.

But you see that-- so now combining this-- so maybe I'll go over here.

So combining this, then we get that if n here is bigger than capital N, then xn minus x is smaller than delta.

But this means that now if they are smaller than delta, then the first part, say, that their images-- so this means that f of xn minus f of x is smaller than epsilon. And that's exactly what we wanted to prove to show that if you are sufficiently far out, then these two things are very close together. So that was quite easy.

Now I want to use these-- but this is like you use this sort of thing for continuity. This is a standard thing to use. It's used implicitly. And again, this one has a converse. And the converse is also very useful. So now let's come back to the extreme value theorem and the intermediate value theorem.

So now let's try to prove-- so using the lemma, we'll show the extreme value theorem and the intermediate value theorem.

And so now first, to prove the extreme value theorem-- so proof of the extreme value theorem. I'm first going to prove that the image-- so first show that the image of this interval here is bounded.

So this is what I first want to show. And so I suppose not-- so suppose not. This means that for each n which is an integer, let's say that it's not bounded from above. To show that it's not bounded from below is similar.

So I'm just going to prove that it's bounded from above. But to prove that it is bounded from below is similar with obvious modifications. So suppose it's not bounded from above. Then for each integer, there must exist x in this interval such that f of x is bigger or equal to n. Because otherwise, n would have been an upper bound.

And so I just pick such a n-- so pick. such n, x-- and let's call it xn. So for each n, you now have xn. And it has a property that f of xn is bigger or equal to n.

Now you have that this gives you a sequence. So you have that xn here is a sequence. It's a sequence in-- sorry, it's in this interval here.

But this means that it's a bounded sequence-- so a bounded sequence so that by the Bolzano-Weierstrass theorem, it has a convergent subsequence xnk.

So you have that xnk-- so this is this subsequence, and it converges to some x. But now this means that by the lemma over there, you have that this sequence then, because the function was continuous, converge to this.

But on the other hand, this here, xnk, is a subsequence of xn. But this means that this number here-- remember that a subsequence just means that this here-- it means in particular that this here would be further out than n.

So you have that f of xnk is supposed to be bigger or equal to nk. But this is supposed to be bigger than n. So these guys here, these here goes to-- you know these here are unbounded. But on the other hand, we're saying that this sequence converges to this sequence.

So any convergent sequence is bounded. So that's a contradiction. OK. So this is like proving first using the lemma, we have now that the image here is bounded from above. The almost exact same proof with obvious modification give that it's bounded from below.

So now you have that the image of this is actually bounded from both above and below. So now this means that if you're looking at the image here-- this here is a bounded subset. So this is a bounded subset.

And so you can look at the sup of the set. You can look at the sup of this set. And I'm going to take the sup-- I'm going to call this capital M.

And now for each little n-- the n is an integer. Then I can look at M minus 1/n. So this is obviously strictly smaller than capital M. So it's not an upper bound.

So this means that for little n, this here is not an upper bound for this set. So this means that there must exist an x so that you have this for some x in this interval here. Because this here is not an upper bound, which means that there must be something larger than that.

So for each little n, you just pick such x. And we call this xn. So xn is in the interval. And we have that m minus 1 over little n is strictly less than f of xn.

Remember that this thing here is, of course obviously less than the sup over all images. So you even have that. OK. But now you see that now you have a sequence here. So now you have a sequence. And so you have a sequence, xn.

And so by the Bolzano-Weierstrass theorem, this sequence here has a convergent subsequence.

This is because this is sequence, the xn lie in this bounded interval. So it's a bounded sequence. So it has a convergent subsequence. And this convergent subsequence, I call xn k, like that. So this is converging to something. And that something, I'm going to call xM. And I then will prove that this has the property that I wanted.

So I have that. And so because f was continuous, then I have that the images converge to the image of this. This is by continuity of f-- and by the continuity of f and the little lemma we proved before.

So you have that. But now you see that this here tells us-- we have this inequality here. But this is just saying that as n goes to infinity, this sequence here converge to M. So we also have that this sequence here converges to m.

This is just because it's squeezed. It squeezed here in this interval. And this here, this lower bound is going to the upper bound. So if you have that this here is converging, then any subsequence would also converge to the same thing. So this implies that for this here, that is a subsequence of these. But this is then converging to the same thing.

So now you see that we have that this is converging to that, but it's also converging to this. We have, by uniqueness of limits, then we have that f of xM is equal to this.

And capital M, again, was the sup. So this means that the sup is actually achieved. You found a point where the sup was achieved. OK. So that proves the extreme value theorem. By one of the two parts of the extreme value theorem, we proved that the sup was achieved. And the proof that the inverse achieved is almost identical, with some obvious changes.

Now for the intermediate value theorem-- so let's try to prove that. So how do we prove the intermediate value theorem? And so proof of intermediate value theorem-- and I'm going to prove, but it's not really-- the general case follows easily from this.

So I'm going to make an assumption. I'm going to assume that the value at a is strictly less than 0 and the value of b is strictly bigger than 0. And then I want to show that 0 is achieved as the image of something.

You can say that sounds like a special case. It is a special case. But once you can prove this, it's easy to get the general case. You can either get it directly from this by modifying the function. Or once you see how the proof of this goes, then it's easy to prove it, to set up what are the different possible cases and then prove using the same sort of argument as we do here.

OK. So this is what I'm going to show want to show, that there exist x between a and b such that f of x is equal to 0. So this is what we want to show.

And again, of course, the function f is obviously continuous. OK. So now let me define into a set. It's actually an interval.

So I define E here to be the sets of x between-- let's call this here. So I want to show that there exists some x. Let's call it x0, just to have something to refer to. I call it x0 because the value's going to be 0.

So I define a set, E. And this is the x in the interval. But I'm now requiring that f of y is less than or equal to 0 for all y which is between a and x.

Now, E here is nonempty. E is nonempty. E here is not the empty set. Why? Because actually, a here-- this is since a here is in E. Because if you take a, then the value is actually strictly less than 0.

And everything that lies in here but lies to the left of a-- well, that's only a right. So this is a kind of trivial statement. So this is not 0. So now you can take the sup of this. So maybe I should make one more remark-- that, of course, not only is it not the empty set, but E here is bounded.

So the sup of E is less than or equal to b. Because all of the elements in E are in this set. So they're trivially bounded by b. So I now can look at the sup. So the sup here is a nice number. And I can look at the sup.

So the claim here is-- or maybe let me just first say set x not equal to that sup of E. And then the claim is that f of x0 is actually 0.

And so now you see that if you take-- right. So now look at the sup.

So for each n, say, integer, if you're looking at x0 minus 1/n, this is not the sup of e. So this means that there exists an element.

For each n, there exists an element, which are called xn. And this xn is larger than this. And the xn is in E. But you see that for it to be an E, then, in fact, this element, the value has to be less than or equal to 0.

So f of xn has to be less than or equal to 0 since it's in E. So since it's in E, then we have that.

But now, of course, the xn's, the xn's, they converge. It's bigger than this, but they are less than the sup of all such guys. So this means that the xn-- so the xn converges to x0. And so this means that f of xn converges to f of x0.

Now this means that this here is-- and these here are all non-negative numbers. This means that this number here must be non-negative. If x is 0, then that's where we want it. So we're done if f of x0 is actually 0.

So assume that f of x0 is then strictly less than 0. But if the value here is strictly less than 0, if this is the case, then there exists-- then if you take-- so here, you have 0.

Here, you have f of x0. You can then take a little interval around it. So take epsilon bigger than 0 such that this thing here-- such that f of x0 plus epsilon, this is still negative.

So I'm going to look at-- so here, I have f of x0. This is actually negative. And I'm taking an interval around it. So this is like an interval. Here, we have f of x0 plus epsilon. And here, we have f of x0 minus epsilon. So I'm looking at this little interval here around f of x0.

And here, I have x0. But now because the function is continuous, I know that there exists an interval. So there exists some delta. There exists delta bigger than 0 so that if I'm in this little interval-- this is the interval from x0 minus delta, x0 plus delta.

If I'm looking at this little interval, then anything here is taken into this thing here. So in other words, I have that if x minus x0 is less than delta, then f of dx minus f of x0 is less than epsilon.

But everything that is in this interval here is still strictly negative. So this means that in fact-- in particular, this implies such that f of x is actually negative.

So now I want to get a contradiction because I know-- so now I have this picture here. I have my x0 here. The x0 was supposed to be the sup of the set A.

So I know that, actually, on a little interval around this x0, everything is-- if I take anything in a little interval around this x0, then the values of f are actually negative. But you see that this x0 was limits or points where everything to the left was not positive.

So it means that everything to the left was not positive. But then even a little bit further was still not positive. But then it wasn't like the sup. Because anything here would be larger, and it still would have the property that everything to the left under f was non-negative.

So that's the proof of the extreme value theorem and the intermediate value theorem using sequences and using that little lemma from before. OK. So now the next thing I want to talk about is another really important concept. And that, to some extent, is a generalization of things we know from the real numbers.

And so this is this concept of a metric space. And so let me just give you a definition, and then I'll give you a bunch of examples. So a metric space is a set, X. So this is the set.

And then it's a map for each pair of points. So let me explain what I mean by this. So whenever you have x in X and another element in this space, then when you have any such pair, then d x comma y, this gives you a real number.

And so this here is what you usually call the distance between x and y. That's the distance between x and y. Now, what properties should this have? It's not just any map from this space across itself into R. It has three properties.

The first is that the distance between pair of points-- and then there are generalization of this. But anyway, so I'll talk about it. That's generalization. Like, if you're working with general relativity, then you would actually not quite work with this definition. You would work with some generalization of this.

But so what properties should this distance have? It should have three properties. The first is that for any x and y, the distance between any two points should always give you a non-negative number. So this here should be non-negative. And if the distance between two points is 0, then those two points should be the same.

So in fact, the distance here is equal to 0 if and only if x is equal to y. So that's the first property. And the second property is-- just their distances are non-negative, if you will. The second property is that if you're looking at, what is the distance from x to y? The distance from x to y should be the same as the distance from y to x.

So this is this. And this is sometimes referred to that it's symmetric. It doesn't matter whether you first-- you begin at x and go to y or you begin at y and go to x. And then the third thing, which is kind of like the nontrivial part of it and the thing that makes it really useful, which is called the triangle inequality-- that if you have three points, x, y, and z, in this space and you're looking at the distance from x to z-- so if you imagine that you have three points-- x, y.

And then you have another point, z. Sorry, no. That's not how I [INAUDIBLE]. So you have three points, x, y, and z. But you think about that you're looking at the distance from x to z. Well, instead of going directly from x to z, if you went over y, then this distance here should be longer than if you went directly.

So this thing here should be less than or equal to the distance from x to y, plus the distance from y to z. And this here is the nontrivial part of it. And this is called the triangle inequality.

Now, what are some examples of this? Let's look at a bunch of examples. The first example is just where x is R. And so you have two points in x in R. And now the distance between them is simply x minus 1.

This is obviously non-negative. If it's 0, that means that x is equal to y. It's obviously also symmetric. And the triangle inequality is this inequality that we're using all the time. Another example is if you're looking at where x here is R squared, say. And then you're thinking about x and y.

You may think about a mesh vector or whatever you want to think about it. So this means that x here has coordinates x1 and x2. And y has coordinates y1 and y2. And now the distance between these two points is just the square root of x1 minus y1 squared plus x2 minus y2 squared.

That's the distance. And this here is often referred to as the Euclidean distance. So this distance that you know. The distance is given by Pythagoras' theorem. And so it's clear the first two property, those are always clear. And the triangle inequality, that's the one that sometimes is tricky to prove. OK. So that's another example of a distance.

I want to give you a few more examples. And so another example is like where you do this in R3. And so you have, instead of points having two coordinates, they now have three coordinates. And you do the sort of same of thing. But now there would be a third term under the square root sign, would be the square of the difference between the third coordinates.

OK. Then let me give you a few more examples. There's one that is, to some extent, simpler than this one here. And it's sometimes useful. But it's not as useful as the Euclidean distances.

And again, these sort of things are generalized. They're an example of something much more general. Like, this here is what's called a Hilbert space, example of a Hilbert space. The next is something which is an example of something called a Banach space. But so the next example I will look at is what is sometimes called the box distance.

And so if space in this case is R2-- you could do it in R3 also if you want. And you take two points here, x and y in R2. And so this here, underline x, has coordinates x1 and x2. And now the distance between these two points would be-- you didn't do this fancy thing with square root, but you did the more trivial thing, where you're just adding.

So you're taking two points here in R2. This is maybe x1 comma x2. This is y1 So y2. The Euclidean distance is simply this distance here. The box distance is where you take this distance here and you're adding this distance here. So this is this here plus x2 minus y2. That's a box distance.

And obviously, this is a non-negative. It gives you-- for any pair, it's non-negative. It's obviously symmetric. It doesn't matter the ordering of these two. Its just switching the order between absolute value sign anyway, so it doesn't matter. And it's also clear that it's 0 if and only if these two points-- it means that the coordinates are the same.

And then, in this case, the triangle inequality is not very hard to establish. Proving the triangle even just for the Euclidean is a nontrivial thing. So that's another example. And I want to give you a few more examples. Here's a sort of interesting example.

Suppose you take a interval-- let's just make the interval from 0 to 1. So suppose you're looking at the interval from 0 to 1. And now looking at-- so this here, this is now your space, x. And so by that, I mean that x here consists of-- so an element in x-- let me write.

An element. In x is a continuous function on the interval from 0 to 1. So this means that if you take two elements-- so if you take one continuous function on the interval from 0 to 1 and another continuous function, what should be the distance between two continuous functions?

Well, the distance here-- so this here is defined to be the maximum of-- so if you're looking f minus g, If f is continuous, g is continuous, this is continuous. And now this is a continuous function defined on this interval.

So the extreme value theorem say that the maximum in-- so this here is continuous. And in fact, even this here is continuous. So the maximum of this here is achieved. So this is the maximum, where x is in 0 to 1. So this is achieved. And this is clearly, again, symmetric. Because of the absolute value sign, it's clearly symmetric, in terms of f and g.

It's also clear that it's non-negative. And it's also clear that it's 0 if and only if the two functions are the same. And in this case, it's quite easy to prove that this is a metric. OK, so maybe I'll just leave that. OK. Now I want to give you a couple of more examples. And then I'll start with example and talk about-- I want to give you a couple of more examples.

And the two next examples, one is-- and the reason why I want to give you two other examples is that-- so you know a lot of things about distances already. And this gives you excellent intuition to proving many things. But you should just keep in mind that a metric space could be very wild.

So I'll give you an example of some strange and directly wild metric spaces. So the first one is kind of strange. And then I'll give you another example, which is wild. So the strange one, so the strange example-- and so this is like if you're looking at the integers. So the space is now the integers.

And so you want to look at, what is the distance between two integers? Now, of course, your intuition would say that if one integer is really large and the other one is not very large, you would think that the distance was large. But it doesn't have to be large. So I define this distance to be 1/n minus 1/m.

This here gives you so the absolute value of the difference of 1 over each of them. This here gives you a distance. And in particular, you see that if both n and m are really large, then in this metric, they're very close together. It looks like the distance is very small.

Now, instead of actually proving that this thing here is-- so you could try to prove-- so as usual, to prove that this is a metric, the first and second property, the property that is non-negative, that's obvious. It's also obvious that it's 0 if and only if n and m are the same. It's also obvious that it's symmetric.

The triangle inequality might be not quite as obvious. It's not very hard in this case. But it actually is a triviality because you could think about this one here as a map. Suppose you take a map from the integers into the real numbers. And the map is just that you take-- the image of an integer is just 1 over it.

If you think about the usual distance over here-- and so the usual distance. By that, I mean that the distance between x and y is just this. If you think about this, but now thinking about that things here-- and so you think about the usual distance. But you're now restricted to not just all real numbers, but just things that are 1 over integers.

On that space, it gives you a metric. But then you can just think about that this gives you another way of labeling things. And so it's kind of a triviality that it is a metric space. But distances looks a little counterintuitive. Because if you have two very large numbers, then actually, the distance between them is small, even though those two large numbers could be very far apart.

So the last example I want to give you, that's a example of a wild metric. This is one wild. And so, essentially, all of these occur naturally.

And so this example is what's called the French railway metric, which is maybe not a super flattering name but probably accurate. So the thing is that if you travel in France-- so if you imagine that this is France here. Here, you have Paris.

And if you want to go to the French Riviera, which is maybe down here-- so two cities on the French Riviera that are incredibly close together is Cannes and Nice. This is Cannes. And here, you have Nice. They're very, very close together. But if you were going to go with a French railroad, then you would have to go to Paris first. So this here would take you maybe 20 minutes by train if there was a train. But you have to go to Paris first. And that's really far.

The good thing is that the French have this TGV, which is this incredible fast train, but it still would take many hours. And so that's the idea of the French railway network, that the space here is R2. And if you take two points, x and y, in R2, then what is the distance between this pair?

So the distance between the pair is-- the distance between x and y is-- if x and y lie on the same-- so imagine that this here is Paris. And so just think about Paris as being the origin.

So if x and y lie on the same line-- so lying on the same line just means that x here is equal to some constant. If this here is y, then x is like a constant times y. Or if one of them was 0, then you could have that y is equal to a constant times x.

So this just means that these x and y lie on the same line. So if they lie on the same line-- so if x here is equal to c times y or y is equal to c times x for some real number, if that's the case, then it's just that the distance between them, it's just like this usual distance here between them.

But if they don't lie on the same line-- they may lie on two lines that are very, very close together-- then the distance here would be the sum of these two otherwise. OK. So that's the French railway metric if I didn't have some sort of strange properties, somewhat counterintuitive properties.

The next thing I want to talk about is convergence in a metric space. So a lot of things we have done generalizes to general metric spaces. So first, let's talk about, what is a sequence in a metric space? So now you have a metric space here. So this is a metric space.

And a sequence xn is a sequence. So maybe I should say it like before, that if you take a map from the integers into x, this is a sequence. And as before, the image of an integer, you would usually denote it like this.

So this is the same definition that we use for the real. So this is a sequence. A subsequence of a given sequence-- well, then you have another map, g, from the integers into the integers and is supposed to be strictly increasing.

And then you're looking f of g of n. And you think about this. You will typically write it like this, f of g of k, say. And you would typically write this as nk right. This here is a subsequences. It's just exactly like the way we define it for the real numbers.

Now, that a sequence is converging in a metric space, that makes also sense. So suppose that you now have a general metric space, and xn is a sequence in X.

Then we say that xn converges to x. So x here is another element in X if for all epsilon greater than 0, there exist a capital N so that if little n is bigger or equal to capital N, then the distance between xn and x is smaller than epsilon.

This is what it means to converge. Now, similarly, the concept of being a Cauchy sequence, this also extends to metric spaces. So a Cauchy sequence in a metric space is the following.

So a Cauchy sequence-- let's call it xn. It's the following, that for all epsilon greater than 0, there exist capital N such that if little m and little n are bigger than this capital N, then the distance between them is smaller than epsilon.

This is what it means to be a Cauchy sequence. Now we have a theorem that is like the same theorem. The proof is like the same proof, but we'll just go over it. So we have a theorem. Let's say that if you have a-- so theorem is the following.

So, again, X comma d is a metric space. And if xn is a convergent sequence in X, then xn also a Cauchy sequence.

Now, how do we prove this? So prove it really is kind of the same as we did before. So prove-- so we have that xn converges. Let's say that the limit is x.

So this means that given epsilon greater than 0, there exist capital N such that if little n is bigger or equal to capital N, then the distance here-- and I'm using now, in the definition of convergence, not epsilon but epsilon over 2.

So I have that. But this means that therefore, if little m and little n, if they are both bigger than capital N and you're now looking at the distance from xm to xn, well, by the triangle inequality, this thing here, it's worse. Or it's certainly not better if you go over this other point.

But you see, this one here and this one, each of them is smaller than epsilon over 2. So the difference here is bounded by epsilon, as it should be. So this proves that if you take a convergent sequence, then it is a Cauchy sequence.

Remember that Cauchy convergence theorem was saying that in R, if you have a Cauchy sequence, then it was also convergent. But that actually fails. This is not true. And it's some important property of a metric space if it has the Cauchy convergence property.

So let me give you a very simple example. So suppose that your metric space-- so the space here, let's just take the interval from 0 to 1. And what is the distance between two elements? So x and y are in this integral from 0 to 1.

What is the distance? I just take the usual distance. But now if I'm looking at this sequence here, xn, which is 1/n, this thing here is a Cauchy sequence. It is a Cauchy sequence.

Because when n and n are really large, then both of them is close to this endpoint. But they don't converge because-- in the larger space, R, they converge to 0. But 0 is not part of this space.

So they don't converge to anything in this space. So this is an example-- so it's a Cauchy sequence, but xn does not converge in x 0. This is because 0 is not part of this interval. And 0 would have to be the limit-- yeah?

STUDENT: Is that the only type of counterexample?

TOBIAS COLDING: No, no. But this does provide an important example of counterexamples. Because then it becomes like a question. Can you make the space Cauchy complete? And sometimes you can. Sometimes you can't. But that's a good question. But it seems utterly trivial example, but it's not. It does show you this important part, that it's often useful to have the Cauchy convergence property. And so can you enlarge the space to get that property?

OK. Let's talk about what it means for a function to be continuous also. So now I want to think about-- so I want to think about continuity for general metric spaces.

And one can do this even more generally. But let me settle on something in between something more general and what we already have. So I have a metric space here. And I'm thinking about that I have a map from this space into R. And then I want to answer, what does it mean for this function here to be continuous?

So f is continuous at some point x0 in this space if for all epsilon greater than 0, there exist delta greater than 0 so that if the distance between x and this x0 is smaller than delta, then the images is epsilon close together.

So you see this is a kind of obvious generalization of how we define continuity at a point for when x was part Of r. And so, of course, the function is-- so this is a continuous in a point.

And so f is said to be continuous if it is continuous at all points. So this is what it means for a function to be continuous.

OK. Now let me just give you an example of something like this. So remember, before, we looked at the continuous functions. So we looked at where the space was the continuous function on the interval from 0 to 1.

And we said that the distance between two continuous functions defined on the interval, this thing here, is the max of the differences here, like that. And that's quite easy to see, but it's a good exercise to do. It's quite easy to see that this is a metric.

So this is a metric space. So now I can think about-- so now here's an example of a function. This is a function that is like evaluating. So you see this getting a little bit abstract because now I'm thinking about a function on-- so I now looking at the function-- so from x into R. But in this case x here is the space of continuous function on the interval.

OK. So my function is now supposed to be defined on a continuous function. And I just take it to be equal to the value of this continuous function at the initial point in the interval. OK.

And I claim that this function here-- so the claim is that this capital F is continuous. So again, it's continuous in the sense that you have this general metric space. And this here-- just so that it lines up with the example.

So we have this general metric space. And we have now a map from this general metric space into R. This is definitely a map from this general metric space into R. And I claim that this is continuous in this sense here. And that's also quite easy to prove.

So let me just make one observation, that if I'm looking at this thing here-- so this is the image under this map in one continuous function. And then I subtract the image in another continuous function. And now I'm looking at this difference here. But this thing here is nothing but f of 0. And this is nothing but g of 0.

But you see that this is obviously less than or equal to the maximum of all x in this interval of f of x minus g of x, like that. But that is the distance between these two elements of this general metric space.

And so you see that this here is an example that if-- so given epsilon, I'm going to pick delta equal to that epsilon. Because if these guys here are now epsilon close by this inequality, their images is also epsilon close. So this map is clearly continuous.

OK. So I'll stop here. So there's a little lemma in the notes that we didn't go through. And that lemma is just the lemma that I had over here, that if I had a continuous function and I have a sequence that converges to a limit, then I could exchange the order of limits. And the same proof works in a general metric space. OK, great. Yeah?

STUDENT: What kind of properties do you need to generalize [INAUDIBLE] space? Is it some kind of closure under addition?

TOBIAS COLDING: Yeah, yeah. Right. That's right. So addition is something about a field. And so for these spaces, most of the things I looked at here, some of it does. Like, the continuous function has some structure. The space of continuous function has some structure. It's not exactly a field, but it has some of those properties, some of these algebra. It's like an important example of something called a sea star of algebra. So it have this algebraic structure.

But it's not the algebraic structure you need in order to be completeness. It's not that. Yeah. And yeah, I think it takes us a little bit far afield. But it's the same-- you see this sort of property also in-- like, metric spaces, extremely useful. You see it all over the place. One of many places where you see it is also in general relativity. And in general relativity, the space you're looking at are typically not complete.

So there's something about that, that you reach something in finite time. You reach it. You're going. And this is like the same thing, that you reach the end of things. Anyway, but it's not the algebraic property that is used for this.