[SQUEAKING]

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PROFESSOR: So we're going to prove the Picard-Lindelof theorem. And we'll see. It may take us-- we may need to continue on it also on Tuesday. And then when we're done with it, we will just do review on Tuesday. But let's see how far we get. So the Picard-Lindelof theorem is the following theorem. And it says that it's an ODE, so it's what's called an ordinary differential equation.

And so again, a differential equation is something about an unknown function and its derivative. So a differential equation just means it's an equation. So this here means that it's an equation in an unknown function and its derivative.

Ordinary, just as we talked about last time, ordinary refers to that is just of one variable. And so the ODE that the Picard-Lindelof theorem is about is-- so the unknown function is y, and the ODE is this that we want the derivative of this unknown function to be equal to a function of the unknown function, plus another function that just depends on x.

So y here is the unknown function, and y is a function of x. x is a real number. And it's just a-- yeah. So this is the differential equation, and then we have some initial condition. So at zero, we want it to take a value "a." So this is the differential equation.

And what we assume here is-- so f and g are function on R. And you should think about those as being known, and they are-- and f here is differentiable. And the derivative is continuous, and g is just continuous. Doesn't need to be differentiable.

And the Picard-Lindelof theorem is-- so again, the Picard-Lindelof theorem is the following, that given f of g as above and-- sorry. And a, a was this constant. It's just a real number, as above. There exists a delta greater than zero, such that-- and a solution. y defined on the integral from minus delta to delta and a solution y to the ODE.

And so again, meaning that the solution-- that means that at zero, it takes value a, and then is satisfying that equation. And moreover, and I'll get back to that, but we won't. It's quite possible I won't get to that today. Moreover, solutions are unique.

And so this is like a standard problem, in when you're looking at differential equations, you have a differential equation that may describe something in nature. And you wonder, does this equation have a solution? And if it has a solution, are solutions unique? So this is what this answer in this case.

And the interesting thing, or part of the interesting thing, so this is a fundamental question used all over science and engineering. And one of the interesting things is that it uses the proof of this uses metric spaces. And so the proof has been used-- various versions of The proof have been used to prove existence and uniqueness of various differential equations. So the proof uses the theory of metric spaces, which is sort of surprising because it seems to have nothing to do with metric spaces.

And so it uses, in particular, the metric space that it's using is the continuous functions on some interval here. And on the space of continuous function, we have the metric that we talked about, the metric. So if f and g-- so f is bad because it's the n. g also. Let's call it h1 and h2. Suppose that these are continuous functions. h1 and h2 are continuous functions on an interval like this.

Then the distance is this that we have considered various times. This is just the maximum over x in this interval here of f of x, of h1-- sorry. h1 of x minus h2 of x, and then the absolute value of that. So this is the distance on the space of continuous functions that we have looked at a lot.

And it's actually not quite this space, as I mentioned last time. It's a ball in this space that it's using. So what do we need about this metric space? So one thing we've proven a couple of lectures ago is that the space here of continuous functions with this metric here is Cauchy complete. So we're going to use that.

Another thing we're going to use is this notion of contracting map. So another tool will be that of a contracting map. And remember, this is a number of lectures ago.

We talked about a contracting map, but we need a more general notion of contracting map, because when we talked about it, we talked about a map from R into R. And then we used this notion of contracting map to finding zeros using Newton's method. But here we're going to use it for something different, and so we need a more general notion.

So suppose so a contracting map uses the general version as opposed to the version that we talked about already that was for the line for R. So contracting map. Sorry. And so you have a metric space x, d. It could be like this, but it also just could be any metric space.

And then you have a map from this metric space into the self. And T here is said to be-- T is said to be contracting. If there exists a constant c less than 1, such that if you're looking at the distance between the images, then they are squeezed together. So this distance here between the images under the map is less than or equal to this constant that is less than 1 times the distance before you apply the map.

So this means that under the map, things get closer together. So that's a contracting map. And so now, we prove the theorem. When we did the Newton's method, we proved a theorem, but we needed about fixed points. And we need a generalized version of that. So if you take a map just in general from a space to a space, then a fixed point, a fixed point is a point x in the space. So that T of x. So the T leaves x invariant. So T of x is equal to x. That's a fixed point.

And so now, if you have a contracting map, so if T here is contracting, then T can have at most one fixed point. Why? Because suppose that x-- so if x and y are both fixed points, well, then you have that T of x is equal to T of y. Sorry. T of x is equal to x, and T of y is equal to y.

And so now you have that the distance between T of x. Let me make a little more space. Then you have the-- actually, we'll just continue down here.

So that the distance between T of x, comma, T of y-- the map is contracting, so this is less than or equal to this constant times this year. On the other hand, they're both fixed points. So T of x is x. This has x, and this has y. So if we write this out, so this is x comma y. And so you see that this thing here is some constant less than 1, strictly less than 1, times itself. So this means, of course, that this number here, this non-negative number must be zero. And so this, of course, implies by the actions of a metric space that x is equal to y.

So a contracting map has at most one fixed point. Now, the other thing is that if you take a metric space-- if you take a metric space, so this here is Cauchy complete. So now I'm going to take a Cauchy complete metric space. So it could be continuous function on some closed interval, and that's how we will apply it, but it works more generally.

So we have this fine theorem. So if this here a Cauchy complete metric space, and T from x into x is a contracting map, then T has a unique fixed point. We know it only has at most one, so this statement really is that it has one right. So it has a unique fixed point.

So let's try to prove this. And you might remember. And if you don't, that's fine. But you might remember how we used it in the Newton method because the Newton method was like-- it was a version of this on R. So you had a map from R to R, and to find the fixed point. And so you wondered what you were there, and your Newton method was something about finding a point where a function vanishes.

And that was set up as a fixed point theorem. And so what you did is you made a good choice, I mean, a choice that looked reasonable. This is not exactly where the function vanishes, but something that looks close to where it vanishes.

And then you iterate. You took the image under the map and you iterated this. And so it's the same we're going to do here. And this time, we don't even need to make a good choice. So we just take a-- we just take-- let x here be any point in the metric space.

So you just fixing some point in the metric space. And now you're looking at-- so you're looking at the sequence x, T of x, and then you apply T to T of x. So I write this as this. And then you apply T to T squared of x, so that's like this. And so you keep applying T more and more times, and so this gives you a sequence, and so on. So this gives you a sequence.

And now the claim is that this sequence here-- so I claim that this sequence here converge to-- so it converge, and the limit here, this here, is a fixed point for T. So that's the claim. Now, so we wanted to prove first that is converging. So we're going to prove first-- so we'll first show that Tn is a Cauchy sequence.

So now let's look at-- so we're supposed to look-- if you're looking at a Cauchy-- if you try to prove that something is a Cauchy sequence, then you're looking at the distance between elements in the sequence where both of these elements is sufficiently far. That's what you're supposed to look at. And so let me write the m here.

So I can think about one of them as being further out than the other one. So I'm going to think about m here as being n plus some number k, some integer k. And so I'm interested in estimating the distance between Tn of x and Tn plus k of x.

Now, in order to do this, let me start with something simpler first. Let me look at the distance between T of x. Let me look at the distance between Tl of x and Tl plus 1 of x. So two consecutive elements in the sequence.

Now, I can think that this thing here, well, this is equal to the distance between T of Tl minus 1 of x. And then T of Tl of x, like that. I can write it like that. But now, of course, so you see here there's some elements in x in the space x. Here's has another element. They both have T applied to it. T is contracting, so you gets c, Tl minus 1, x, Tl of x.

So you see that if you take two consecutive elements and you're taking the distance, then it's bounded by this constant c times where you have 1 less-- where you have applied T 1 less time on both. These are, again, consecutive elements, but the highest one is 1 less. So now, if you iterate this, so iterating this, so let's see. Maybe let me wait with iterating. Let me just look at what we need.

So we have this thing here. And this here I can write as-- so here you see, here I can write this as T, the distance T, And then Tn minus 1x, and then T, Tn minus 1 plus kx, like this.

And so here I can use the same trick that you guys see, and then d, Tn minus 1x comma Tn minus 1 plus k of x. And this one here. I can keep going like this. I can keep going until there is the-- I've used up all my ends. And so I can keep going. And so this is less or equal to c to the n. And then the last one I get is, I get x, and then I get Tk of x. I want to show that this thing here becomes small as long as n-- remember, k is a positive integer as long as n is sufficiently large.

So this is a hopeful sign, because when n gets large, this here gets really, really small. But I still need to estimate this here. Now, so how do I estimate it? By using what I did over here.

So you're looking at-- so you have the distance between x and Tk of x. This thing here, I can use this triangle inequality. So I can say that this is less than dx T of x plus dT of x, T squared of x plus dT squared of x, cubed of x. And then all the way up to d, Tk minus 1 of x, Tk of x. So this is just the triangle inequality.

Now, this may seem rather bad estimate, but let's see how bad it is. So now, this first thing here, I'm just leaving it. I can't do anything about that. So I need to just take that. This one here, so I'm using this with between two consecutive. So in this case, it's just bounded by c times dx, Tx. And the next one here, this one here is bounded by c squared, dx, Tx. And the next, and so on, up to this one here is bounded by a c, k minus 1, dx, Tx.

And so you see that now, this here factor out. So this here is this factored out. And then, in this bracket, you get 1 plus c plus c squared, plus all the way up to ck minus 1. But this is a geometric series. This is a geometric series. And this is certainly less than when I take-- remember c is strictly less than 1.

So it's a geometric series. It's a convergent geometric series. So this here is less than dx, T of x. And then this sum, well, it's less than when you're summing it all the way to infinity. So this is less than 1. This is the sum of the geometric series. C is strictly less than 1. C is positive, and it's strictly less than 1.

So/ you see now that we have-- so we have that. So what we were interested in is, we were interested in saying that, as long as I choose little n sufficiently large, then this thing here must be very small. And we know already that it's bounded like this. And now I fill in this, and you see this here.

So this means that this is less than cn. And then what you get here is 1 minus c, and then times dx T of x. x is fixed, and so you see that when n, little n, is sufficiently large, then this whole thing here is as small as you want. So this means that what we have proven here is that this sequence here is a Cauchy sequence. So we have proven that.

Now, of course, this is where the assumption comes in that this metric space is Cauchy complete. So since x, this metric space is Cauchy complete, and this sequence here is a Cauchy sequence, it is convergent. Yeah.

STUDENT: I'm sorry. Just back to the end of that repeat.

PROFESSOR: Yeah.

STUDENT: Are you able to explain again how we got that feature, or how we were trying to-- did we find a bound for the dx, Tx, or we're just saying that--

PROFESSOR: So I'm saying that x is fixed. You pick any point you want, and I claim that if I-- it may be a very bad candidate for the fixed point, but it doesn't matter. It's OK. It may just take you longer to get to the fixed point. But you fix x, and then you're looking at the sequence.

And then I claim that this sequence will be a Cauchy sequence. And so x is fixed, and so we have this inequality here. And what we need is that when little n is sufficiently far large, then for all k, k is a positive integer, This is further out than in. Then for all k-- and this should be independent of k-- this thing here small. But this is just--

STUDENT: Doesn't that get rid of-- the whole thing from the previous step was now it's not a Tk of x.

PROFESSOR: Yeah, so since x is fixed, x is fixed. Tx is fixed. So this is just a constant. So right. So what we have proven is that-- so we have so we have this Cauchy complete metric space. And we have this sequence. X was fixed again. That is a Cauchy sequence. And so since this metric space is Cauchy complete, this is convergent, and I can denote the limit. So I just called the limit x infinity. So we have this.

Now, I want to show that-- so the next claim is that x infinity is a fixed point for T. So if I'm looking at this sequence here, suppose I take the sequence. And suppose I take T of this sequence. So if I take T of this sequence, this here is nothing but the next element in the sequence. So this one here will still converge to x infinity, because it's just shifted by 1.

But then on the other hand, so now, if you're looking at-- so you have that. If you're looking at T, so we want to prove that-- so now, if we're looking at T of x infinity, this distance between T of x infinity and Tn plus 1 of x, suppose I'm looking at this distance here. Well, this here I can think about as the distance between T of x infinity and T of Tn of x.

But here there are now both of these elements. You can think about this as just some elements. Apply T to it. Here there's Ts applied to that. So this is less than the distance x infinity to Tn as a constant c less than 1 in front. So like this by the contracting map property.

But now you see that as n goes to infinity, this thing here goes to zero. So this means that this thing here converges to this point. So you have that-- so you have that-- you have that if you're looking at-- if you're looking at-- again, as n go to-- as n goes to infinity, since Tn of x converges to x infinity, then we have that the distance-- this is the equivalent to that the distance here goes to zero. And then by this inequality, it means that the distance here Tx infinity and Tn plus 1 of x goes to zero.

And so this means that this is just some fixed point. It means that this sequence here-- so this implies that this sequence here converts to T of x infinity. But on the other hand, we already observed that this sequence also converges to x infinity. So it converges to this and it converges to that, but it can only have one limit, so those two things are equal. So you conclude that x infinity is equal to T of x infinity. And so x infinity is a fixed point.

So this proves this theorem right. And again, we already knew that the limit was unique. So now, how is this related to the Picard-Lindelof theorem? And so it's related in the following way.

And again, this is something that has been used to solve a lot of differential equations. And so we need to define a space in a map. The spaces initially, but we have to restrict it. But initially, let's think about some continuous functions on some interval. Let's start with just continuous function. But it's not quite going to be the space. I will have to look at a smaller-- Yeah.

STUDENT: Question. Why is it that you didn't need to know the need before to a good starting point?

PROFESSOR: That's a good question. So that's right. So this comes with this Newton method. So we used the-- so we used this contracting mapping for R in the Newton method to find a zero. And to some extent, there was a great idea. Newton was a pretty famous guy for a good reason.

But in a way, this is more ingenious, I think, because it's like metric space. What the hell does this have to do with differential equation? And the reason why in the other one, we had to restrict it was that the map we defined wasn't contracting everywhere. It needed things to be pretty close initially for there to exist a c strictly less than 1. So that's where it came in. It wasn't really because if-- yeah.

So now, again, so we need to relate this. That seems to have nothing to do with differential equation. We need to relate it to the differential equation. So again, what is the space, the metric space that we're going to look at? And again, it's not exactly this, but it's just a subset of it. So it's this space, and with the usual metric that we have talked about. That is going to be the metric space.

Now, what is the map T? And so the map T is-- so T here is going to be a map from continuous functions into the space of continuous functions. And is going to be defined as follows.

And I'll come to this in a second. So T of x, T of y. So think about y as being a continuous function on this interval. And T of y is now equal to a plus, and then the integral, and then integral. So this is a-- so T of y, y is of an element here. This one here is also supposed to be an element here, so this is a function.

So we want to find out what the value is at x. And this a, this constant a plus the integral from zero to x, and then of f of y, plus g of x. And this here is a function of x, also, so like this. And I shouldn't write x because I use now x up here. So I write it like that. d of x. That's how the map is going to be defined.

So again, if you take-- so y here, so y is going-- so y is a continuous function on this interval. And I will have to restrict it, and this is exactly when you're asking what in Newton's method, why did we have to pick this good choice. It's exactly like this that this map is not contracting on this whole space. So we have to-- in a way, we have to do the same stuff. We have to make a good choice first. And so that's why we have to restrict the space. And it actually is-- part of the beauty is that it's pretty much optimal, this stuff, this sort of general principle. We can come back to that.

So this is the map T. And so the first thing that you observe is that T-- so this gives you this function here. Remember that so if y is continuous, g is continuous. Well, if y is continuous f is continuous. Everything here is continuous. And so this means that by the fundamental theorem of calculus, this function here is actually even differentiable. And the derivative of this is just a constant. So this function here is differentiable, let alone continuous. So it's definitely in the space and the derivative of this map.

By the fundamental theorem of calculus-- so by the fundamental theorem of calculus, the derivative is just what is inside here. So this is just f of y plus g of x. And if you want to write x in, then it should be like this.

And then one more thing you observe, which is also immediate, that if you take a continuous function like this, and now you evaluate this, now you evaluate this at zero. But if x is zero within this integral, it's over just a point. So this here is zero. So this means that the value here is a.

And so now, the next thing you observe is that-- so the next thing we observe is that if you have a fixed point. I think this is another nice thing is that the way it's set up is that you just need to use continuous functions. But of course, if you want a solution to this, it means that the function must have a derivative. Yeah.

STUDENT: So how do you know that [INAUDIBLE]?

PROFESSOR: Yeah we don't know that yet. That's a major point, actually. And again, just like in Newton's methods, we have to restrict where we're looking at it. Yeah. But the first thing I want to observe is that if y is a fixed point, well, then T of y is equal to y.

And so what does this mean? Well, the derivative. So it means that if this is the case, then we have that T of y is of this form. So it has a derivative, and that derivative is this. So this means that y would have it, because y is equal to this. So y is differentiable, and the derivative is equal to this.

And we always have-- whether it's a fixed point or not, we always have that Ty of zero is a. And since it is a fixed point, this is also this. So you see that if it is a fixed point, then it is a solution to the differential equation.

Now, then the point comes in. The key point comes in with proving that the map is contracting. And so now, so we need to prove-- need to show that T is a contracting map.

So set. I have to define a number of things, because again, I have to-- T is not going to be contracting on the space of continuous functions. So I have to look at some subset of continuous function. And the right subset is just a ball. It's its a close ball. So it's going to be-- so we're going to-- maybe I'll just say first that we're going to restrict T. So T here will be restricted now to-- we will think about it as a map on some ball into itself. But it's not at all clear that this map takes a ball into itself.

And what do I mean by ball? So I have this bar up here. So I just means that this thing here is going to be the function y that is continuous. And then it's not quite going to be this either. I have to restrict. But I'll get to this in a second. So at the moment, you can think about it as being continuous function on the interval, so that the norm of y at each point x is less than or equal to R. And this bar here refers to that equal is also allowed.

Now, this here is really like the ball. These are functions. If you're thinking about the function in the space of continuous function that is constant zero, then these are the functions whose distance-- this is the same as the space of continuous functions whose distance to the constant function zero is, at most, R. This is the zero function. It's less than or equal to R. Sorry. This is getting a little smaller over here.

And again, it's going to be-- I will have to restrict the interval here to make sure that the map is contracting. But first, I want to define this radius here. And so the radius is-- I'm setting R here equal-- a was just a constant. I take the absolute value of a and I add 2 to it. Then I define two other areas, and there's no way around it. You have to define. We will require to define several constants first.

L1, capital L1, is going to be another constant. It's going to be the maximum. And then you take off over z, where z here is less than R, and then you take the value like this. Remember, f is continuous. So looking at where the norm of x is less than or equal to some constant, then this, the maximum is achieved on this. So this is just some real number, positive real, non-negative real number.

And then I define L2 to be the maximum of where I-- of maximum on this unit ball here, and where norm of x is less than 1 of g of x, like that. And now, what I want to do is-- so the first step is-- so the first step is the following.

So first step in the proof is to prove the following, that there exists delta zero. This is not the delta that I will use eventually. I would have to restrict it further. There exists delta zero, so that if you're looking at this space, if y here is in the space of continuous functions on this interval here, then T of y-- sorry.

If y is in this here, and y is in the unit ball-- not in the unit ball, then y here is less or equal to R. So it's a continuous function on this interval, and the norm is less or equal to R. So this means that so this is, of course, the same as saying that y is in this ball here when I'm now thinking about this as a space. But you can forget about that if you want. It's the same.

And I want to show that T of y is in the same space. So T of y. So I want to show-- so I want to show that T of y at each x here, the norm is less or equal to 1. Then if I have that, I already know that this function T of y is continuous. It's actually even differentiable. But it's, in particular, continuous. And if I have this, then I know that this space is taken into itself.

So now, how do I do that? So that's where I need to restrict the interval that I'm looking on. And again, we'll come back to it, but there's a good reason that we have to restrict. Otherwise, it couldn't possibly be true. Yeah.

STUDENT: To be contracting, do we need it to be strictly less than 1?

PROFESSOR: So remember, contracting means that there exists a constant c strictly less than 1. This here has nothing to do with the constant c. This here is just the space that we're looking at the map on. And the reason why I make here-- so again, so this has nothing to do with that c. But the reason why I want this less than or equal to is that I want to have that the space is Cauchy complete. Sorry. So let me just try to make that clear.

So to get a fixed point for a contracting map, you need a space that is Cauchy complete when we prove this for the metric space. We have a good candidate. We have the candidate of continuous functions on some closed interval like this. This is Cauchy complete with the usual metric. But I'm not going to look at that space because my map is not actually contracting on the space. It's contracting on a subset. And so I need that subset. So in the end, I'm sort of not working on this, but I'm working on a subset. And I need that subset to be Cauchy complete.

So basically, what I have is that this here is the giant metric space I start with. Then I'm looking at a ball in this. And if I take the closed ball and it happened to be at the origin, but it could be just at any point. If you take a metric space and you're checking something like this, if the metric space is Cauchy complete, then this subset is also Cauchy complete. But this require I have less than or equal to-- because I also could kind of converge to something where you got equality. So that was why I do it like that.

So let's see. So we are trying. We have already defined this radius, capital R. And then we have these two constants, L1 and L2. And what I need to prove is-- the first step here is to prove that if-- I'm going to look again.

So the first step is that I need to restrict. So I need to restrict. So I'm going to look at a subset of these spaces of-- this is a subset of the continuous functions. And I need to prove that if y is in here, then the image is in the same space. And so this is the statement that if y is continuous, the norm of this function is less than or equal to R. So at each point, each x, the absolute value of y of x is less or equal to R. Then you have the same for the image under T.

We already know, again, that the image under the T of y is continuous. And so what I can do is I can restrict my interval. So I will restrict the interval as follows. So I'm going to pick delta zero to be the minimum of 1 over L plus 1. L is non-negative. In principle, it could be zero. I don't want to worry about that, so I just add 1.

And L2 could also be zero, although it is pretty unlikely that any of them are zero. But just to be safe, I'm just adding so I don't divide by zero. So that's why.

And the first thing is that I'm-- and then I'm going to look at the interval from this. And this is not going to be my final. I will have to do some further restriction. But I'm going to look at this interval initially. And the 1 here, it just comes in that I want this here to be a subset of this unit interval. So now, let's try. So now we are in good shape here.

So now, let's look at Ty. And we need to estimate this at each x. And we want to say that the norm is less than capital-- less or equal to capital R.

Well, let's just look at what this is. This thing here is equal to this function. T of y is defined to be a plus the integral from zero to x of f of y of s, plus g of s, ds. And then I have to have an absolute value like that. And now I can just use the triangle inequality, of course, trivially. I get absolute value of this, and then plus, and then the integral here, like that. Oops.

And then I can just take this absolute value in under. So I get less than or equal to a plus, and then the integral, and then the absolute value of f of y of s. And here I can also take the absolute value using the triangle inequality on each individually.

So I can write it like this. Now, this is where I use-- so s here will have to lie between zero and x. And so x is between-- let me just draw it here. So here I have zero. Here I have delta zero. And then I have-- this is zero, and here I have x somewhere.

So this means that-- this means that so here you have x, but x itself, x has to lie-- x has to lie between delta zero minus delta zero, and delta zero. And s has to lie between zero and x. So this means that s here lies between-- x will lie in this interval, also.

But now you see that-- so now if we're looking at-- so we have T of y in x. Again, remember that x here is less than or equal to delta zero. So we have that this thing here is less than or equal to a plus, and then the integral from zero to x of f of y of s, plus g of s ds.

And the first thing I observe is that s here is also, just because x was in that interval, you also have that s-- because x is in that interval, s is also in this interval. The second thing I observe is that I'm assuming that the function that I start with at each point have no less than R. Now, if I'm using these two things, then I have that-- so this means that g of s, this one here, this is how we defined L2 because this interval here is contained in the interval between minus 1 and 1. This here is contained in the interval between negative, minus 1 and 1. So we have that g of s is less than this L2, less than or equal to L2.

The second thing we observe is that because we have this here, for each s, we have this. Then it means, you see, by the first-- by the definition of L1 that f of y of s, this thing here, is less than L1. L1 and L2 are now fixed numbers. And so you see that this second thing here, I can bound this thing here by-- so this is less than or equal to a. And that should have been an absolute value. Somehow I lost it when I copied it from this line. a, absolute value of a.

And then on the first one here, the length of this interval, but that is, of course, bounded by delta zero. So this is plus delta zero, and then the maximum possible value of this, that would be L1. And then likewise, the length of this interval you're integrating over is at most delta zero. And this one here cannot be larger than L2. So you have this. And now, remember what delta zero was. And I will just-- I have just on the bottom, below here. But let me just write it here. This here was what delta zero was chosen to be, the minimum, 1 over L1 plus 1, 1 over L2 plus 1.

So now, we have that this thing here is less than this here. We just keep this thing here. Well, delta is, in particular, less than this thing here. So you see that if you plug in this, so it's like plus 1, L1 plus 1 times L1. Likewise, over here, you're using this. So you're getting 1 over L2 plus 1 L2. And now you see that this thing here is less than 1. This is less than 1. So this is less than a. It's actually strictly less, but it doesn't really matter. a plus 2 like this.

So this means that-- so then, but this here, this was exactly the definition of capital R. And so you have that. And when I copied it over, I also forgot to put absolute value sign here. So you have that if-- so have that if-- so the conclusion here is that if y here is a continuous function on this interval here, and the norm of y at each point is less or equal to r, well, then you have that T of y at each point is less than R. It's actually strictly less, but we don't care about that.

So what you have now is-- so what we now have proven is that we have this map T. And now I'm going to think about this map-- sorry-- on the space of y that is continuous on this interval here. And what you have is-- and also, sorry. And I'm thinking about with the norm at each point. This here is less than R.

And again, this is really a ball in this metric space, but let me just write it out like this. And what you have is that-- sorry. This is the space, and you have that T of y. If y is in this space, then T of y is in the same space. So T of y is also continuous on that interval. And T of y at each point x is less or equal to R. So the map mapped this space into itself.

Now, the next thing we want to prove is that the map is contracting. You see that you need-- you know if this space is not stable. So imagine that you take-- so we're starting with some giant metric space. This is the continuous function, say, on this interval.

And then we're looking at some subspace of that. So these are continuous on a smaller interval. And then we have to find a map. But if somehow the map was taking us out of the space where it's contracting, then we couldn't keep applying the contracting map. It wouldn't anymore be--

And the thing is that you want to just iterate. You want to keep. You want to pick some point, and then you want to keep applying the contracting map. So you want it to stay in the space where it is contracting. And so this is what we did in one. But we haven't yet proved that the map is contracting. And so now, to prove that the map is contracting, we're going to further restrict the interval. And so this is-- so I'm going to set. I'm going to define now one more constant. Sorry about that.

So this is M here. And M is going to be just the max of f prime-- of f prime of z, absolute value of this, where z here is less or equal to R. So this is my-- well, almost the second last constant function.

And now I'm going to define this. So the last constant is going to be the interval that I'm going to work with. So remember, what I'm trying to do is, I have this differential equation. And I want to say that, near zero, there is a solution. But near zero just means that there exists some tiny interval around zero where I have a solution.

And again, if I have time, I also will do it next time. I will give an example where you can see that, in general, you would need to look at a very tiny interval for this to have a solution. Otherwise, it doesn't have a solution on a very large set. So I define it like this. And then I define delta here. So delta is now going to be the minimum of delta zero, the one from before, and then 1 over 2M plus 1.

And now I want to look at the space. So I want to look at T. And then T is going to be a map from the continuous functions. So y here is a continuous function on, now, this little interval.

So this interval here, because you have the minimum of this, this interval is contained in the corresponding interval with delta zero. So it's y in this space, such that the norm here at each point is less than or equal to R. That's the space I'm going to look at. And we already know that T maps this space into itself, because there's really no new statement. If the function is-- so as long as you have this here, and you apply T on it, T, when you apply T on a function, it's defined on the same interval. So it will automatically define on this smaller interval. And we already have this property. So T maps this space here into itself.

And now I want to prove. I want to estimate. So I take two elements in this space. So I have y1 and y2. So you're looking at-- so y1 and y2 are in this space. And you want to look at what is the distance between y1 and y2.

So we want this here, and we want-- what I want to prove is that it's contracting. So we want to show. And I want to prove. I'm going to prove it. My constant c will be explicit. It will be just one half, so that certainly is less than 1. And this is what I want.

So now, let's look at them. So T of y1 minus T of y2. This thing here, T of y1 is defined to be a plus the integral. Sorry. And let me write. So I'm going to-- let me just make it a little more space here. I worry that I'm running out of time, but let's see.

So this thing here, minus T of y2. And I'm just looking at the difference in some x. And again, x here is now in-- x here will be in the integral. You were looking at x in the integral, in this integral. And so I'm just writing down first here what this is. This is the integral from a to x of f of y1, s plus g of s.

And then you have minus. This is what you get for y2. a, so minus this thing here. So this is this difference. And what you see is the first thing you observe is that this a here cancels.

But not only does this cancel, but this term here also cancel. And actually, there was a good reason that there are lower order. And the point here is that by-- the main point is in the unknown function anyway. And it's what's called a non-linear equation. So that's a difficulty. But what you then get when you write down, when you have-- after the cancellation, what you get is f of y1 of s minus f of y2 of s, ds. So you have that.

Now, I will look at this thing here and that thing here. And I'm going to-- and you have to now abstract. So now I'm going to look at this difference here. And then now, the thing is that you have to abstract from it, and you have to think about s as being-- if you think about s as being fixed, then this is just two numbers. You could think about this as z1 and this here as z2. This is z1, and this is z2. So it's like f of z1 minus f of z2.

Now, the mean value theorem tells us that this thing here is equal to the derivative at some intermediate point, let's say, z3, where z3 is between z1 and z2, and then times z1 minus z2. So this is just for any xs, you have this. But this means that-- this means that-- are we are almost out of time?

So this means that when you're looking at the norm of this, f of y1 of s minus f of y2 of s, that the norm here is-- so the norm is equal to f prime at z3 times z1. And z1 I can write in. It's really this. And this is z2.

This thing here, remember, z is some point in between those two points. But both of these points, both of these points here, both this one here and that one here have norm less than R. So this here is-- but now where the way that M was defined, and it should have been on the same board.

So M here was defined to be the maximum of f prime of z, where z was less or equal to 1. And so you see that this thing here where you take the derivative that is bounded by M. So this thing here is less than M y1 s minus y2 of s, like that.

So I'm not sure where we should stop. We over, and I probably need a few more minutes. Maybe we should stop. And it's not a great place to stop, but that's how it goes.

So I will keep you in suspension, and so we'll pick up from here on Tuesday. And so I want anyway to explain to you why, actually, this method, which seems a little crazy in a way because it's bringing in all these things, why it turned out to be basically optimal. And then we'll do a review the rest of the time on Tuesday.