

Practice Midterm 1

No notes, textbooks, calculators, or other materials may be used. Please switch off all mobile phones or other electronic devices.

Unless the problem specifically states otherwise, the rules are as follows: you can use any theorem proved in the 18.100C lectures, and any theorem which is in the body of the textbook. However, you should state the theorem clearly when you use it. You may not use any theorems proved in the homework, or ones which are in the problem part of the textbook (if you want to, you have to reproduce their proofs).

1. Let K be any field, and $x, y \in K$ elements such that $x^2 = y^2$. Prove, **using only the axioms of a field**, that $x = y$ or $x = -y$. Be careful to explain which axiom you use at each step.
2. Let (X, d) be a metric space. Recall that a subset $E \subset X$ is **dense** if $\bar{E} = X$. Prove: if X has a dense subset which is finite, then X itself is finite.
3. Recall the definition of the **p -adic metric** on the set \mathbb{Z} of integers: we fix a prime number p . If $x \neq y$, then $d(x, y) = p^{-n}$, where n is the largest integer such that p^n divides $y - x$. Is (\mathbb{Z}, d) compact? (With proof, of course).
Hint: look at specific examples of subsets $\{x_1, x_2, \dots\}$ such that the p -adic distance between x_i and x_{i+1} becomes smaller and smaller.
4. You'll find below a correct theorem and proof, but where the proof is missing a lot of details. Write out the proof again, supplying the missing details.

Theorem. Let (X, d) be a metric space, $K \subset X$ a compact subset, and $E \subset X$ a closed subset, such that $K \cap E = \emptyset$. Prove that there is some $D > 0$ such that $d(x, y) \geq D$ for all $x \in K, y \in E$.

Proof. Suppose otherwise. Then there are points $x_n \in K$ and $y_n \in E$ such that $d(x_n, y_n) < 1/n$. If the set $\{x_n\}$ is finite, then one of its points is a limit point of E , so must lie in E . That's a contradiction to the assumption that K and E are disjoint.

In the other case, by compactness, the set $\{x_n\}$ has a limit point x . Take some $\epsilon > 0$. The ball $B_\epsilon(x)$ contains infinitely many x_n . Therefore, I can choose one of those so that $d(x_n, y_n) < \epsilon$. The consequence is that $d(x, y_n) < 2\epsilon$. But then x is a limit point of the y_n , a contradiction by the same argument as before. \square

Score: $5+5+5+5 = 20$ points. A score of 12 or higher is considered a passing grade (this has no concrete consequences, it's just for your information.)

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