Lecture 16

4 Multi-linear Algebra

4.1 Review of Linear Algebra and Topology

In today's lecture we review chapters 1 and 2 of Munkres. Our ultimate goal (not today) is to develop vector calculus in n dimensions (for example, the generalizations of grad, div, and curl).

Let V be a vector space, and let $v_i \in V, i = 1, \dots, k$.

1. The $v_i's$ are *linearly independent* if the map from $\mathbb{R}^k$ to V mapping $(c_1, \dots, c_k)$ to $c_1 v_1 + \dots + c_k v_k$ is injective.

2. The $v_i's$ *span* V if this map is surjective (onto).

3. If the $v_i's$ form a basis, then $\dim V = k$.

4. A subset W of V is a *subspace* if it is also a vector space.

5. Let V and W be vector spaces. A map $A : V \to W$ is *linear* if $A(c_1 v_1 + c_2 v_2) = c_1 A(v_1) + c_2 A(v_2)$.

6. The *kernel* of a linear map $A : V \to W$ is

$$\ker A = \{v \in V : Av = 0\}. \tag{4.1}$$

7. The *image of* A is

$$\text{Im } A = \{Av : v \in V\}. \tag{4.2}$$

8. The following is a basic identity:

$$\dim \ker A + \dim \text{Im } A = \dim V. \tag{4.3}$$

9. We can associate linear mappings with matrices. Let $v_1, \dots, v_n$ be a basis for V, and let $w_1, \dots, w_m$ be a basis for W. Let

$$Av_j = \sum_{i=1}^{m} a_{ij} w_j. \tag{4.4}$$

Then we associate the linear map A with the matrix $[a_{ij}]$. We write this $A \sim [a_{ij}]$.

10. If $v_1, \dots, v_n$ is a basis for V and $u_j = \sum a_{ij} w_j$ are n arbitrary vectors in W, then there exists a unique linear mapping $A : V \to W$ such that $Av_j = u_j$.

11. Know all the material in Munkres section $\oint 2$ on matrices and determinants.

12. The quotient space construction. Let V be a vector space and W a subspace. Take any $v \in V$. We define $v + W \equiv \{v + w : w \in W\}$. Sets of this form are called W-*cosets*. One can check that given $v_1 + W$ and $v_2 + W$,

    (a) If $v_1 - v_2 \in W$, then $v_1 + W = v_2 + W$.

    (b) If $v_1 - v_2 \notin W$, then $(v_1 + W) \cap (v_2 + W) = \phi$.

    So every vector $v \in V$ belongs to a unique W-coset.

    The *quotient space* V/W is the set of all W=cosets.

    For example, let $V = \mathbb{R}^2$, and let $W = \{(a, 0) : a \in \mathbb{R}\}$. The W-cosets are then vertical lines.

    The set V/W is a vector space. It satisfies vector addition: $(v_1+W)+(v_2+W) = (v_1 + v_2) + W$. It also satisfies scaler multiplication: $\lambda(v + W) = \lambda v + W$. You should check that the standard axioms for vector spaces are satisfied.

    There is a natural projection from V to V/W:

    $$\pi : V \to V/W,\ v \to v + W. \tag{4.5}$$

    The map π is a linear map, it is surjective, and $\ker \pi = W$. Also, $\text{Im } \pi = V/W$, so

    $$\begin{aligned} \dim V/W &= \dim \text{Im } \pi \\ &= \dim V - \dim \ker \pi \\ &= \dim V - \dim W. \end{aligned} \tag{4.6}$$

4.2 Dual Space

13. The *dual space* construction: Let V be an n-dimensional vector space. Define V^* to be the set of all linear functions $\ell : V \to \mathbb{R}$. Note that if $\ell_1, \ell_2 \in V^*$ and $\lambda_1, \lambda_2 \in \mathbb{R}$, then $\lambda_1 \ell_1 + \lambda_2 \ell_2 \in V^*$, so V^* is a vector space.

    What does V^* look like? Let $e_1, \ldots, e_n$ be a basis of V. By item (9), there exists a unique linear map $e_i^* \in V^*$ such that

    $$\begin{cases} e_i^*(e_i) = 1, \\ e_i^*(e_j) = 0,\ \text{if } j \neq i. \end{cases}$$

    Claim. *The set of vectors $e_1^*, \ldots, e_n^*$ is a basis of V^*.*

    Proof. Suppose $\ell = \sum c_i e_i^* = 0$. Then $0 = \ell(e_j) = \sum c_i e_i^*(e_j) = c_j$, so $c_1 = \ldots = c_n = 0$. This proves that the vectors e_i^* are linearly independent. Now, if $\ell \in V^*$ and $\ell(e_i) = c_j$ one can check that $\ell = \sum c_i e_i^*$. This proves that the vectors e_i^* span V^*. □

The vectors $e_1^*, \dots, e_n^*$ are said to be a *basis of* V^* *dual to* $e_1, \dots, e_n$.

Note that $\dim V^* = \dim V$.

Suppose that we have a pair of vectors spaces V, W and a linear map $A : V \to W$. We get another map

$$A^* : W^* \to V^*, \tag{4.7}$$

defined by $A^*\ell = \ell \circ A$, where $\ell \in W^*$ is a linear map $\ell : W \to \mathbb{R}$. So $A^*\ell$ is a linear map $A^*\ell : V \to \mathbb{R}$. You can check that $A^* : W^* \to V^*$ is linear.

We look at the matrix description of A^*. Define the following bases:

$$e_1, \dots, e_n \text{ a basis of } V \tag{4.8}$$

$$f_1, \dots, f_n \text{ a basis of } W \tag{4.9}$$

$$e_1^*, \dots, e_n^* \text{ a basis of } V^* \tag{4.10}$$

$$f_1^*, \dots, f_n^* \text{ a basis of } W^*. \tag{4.11}$$

Then

$$\begin{aligned} A^* f_j^*(e_i) &= f_j^*(Ae_i) \\ &= f_j^*(\sum_k a_{ki} f_k) \\ &= a_{ji} \end{aligned} \tag{4.12}$$

So,

$$A^* f_j = \sum_k a_{jk} e_k^*, \tag{4.13}$$

which shows that $A^* \sim [a_{ji}] = [a_{ij}]^t$, the transpose of A.