Lecture 30

6 Manifolds

6.1 Canonical Submersion and Canonical Immersion Theorems

As part of today's homework, you are to prove the canonical submersion and immersion theorems for linear maps. We begin today's lecture by stating these two theorems.

Let $A : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map, and let $[a_{ij}]$ be its associated matrix. We have the transpose map $A^t : \mathbb{R}^m \to \mathbb{R}^n$ with the associated matrix $[a_{ji}]$.

Definition 6.1. Let $k < n$. Define the *canonical submersion* map π and the *canonical immersion* map ι as follows:

Canonical submersion:

$$\pi : \mathbb{R}^n \to \mathbb{R}^k, \quad (x_1, \dots, x_n) \to (x_1, \dots, x_k). \tag{6.1}$$

Canonical immersion:

$$\iota : \mathbb{R}^k \to \mathbb{R}^n, \quad (x_1, \dots, x_k) \to (x_1, \dots, x_k, 0, \dots, 0). \tag{6.2}$$

Canonical Submersion Thoerem. *Let $A : \mathbb{R}^n \to \mathbb{R}^k$ be a linear map, and suppose that A is onto. Then there exists a bijective linear map $B : \mathbb{R}^n \to \mathbb{R}^n$ such that $A \circ B = \pi$.*

Proof Hint: Show that there exists a basis $v_1, \dots, v_n$ of $\mathbb{R}^n$ such that $Av_i = e_i,\ i = 1, \dots, k$, (the standard basis of $\mathbb{R}^k$) and $Av_i = 0$ for all $i > k$. Then let $B : \mathbb{R}^n \to \mathbb{R}^n$ be the linear map $Be_i = v_i,\ i = 1, \dots, n$, where $e_i, \dots, e_n$ is the standard basis of $\mathbb{R}^n$. $\square$

Canonical Immersion Thoerem. *As before, let $k < n$. Let $A : \mathbb{R}^k \to \mathbb{R}^n$ be a one-to-one linear map. Then there exists a bijective linear map $B : \mathbb{R}^n \to \mathbb{R}^n$ such that $B \circ A = \iota$.*

Proof Hint: Note that $B \circ A = \iota \iff A^t B^t = \pi$. Use the Canonical Submersion Theorem. $\square$

Now we prove non-linear versions of these two theorems.

Let U be an open set in $\mathbb{R}^n$, and let $f : U \to \mathbb{R}^k$ be a $\mathcal{C}^\infty$ map. Let $p \in U$.

Definition 6.2. The map f is a *submersion at p* if $Df(p) : \mathbb{R}^n \to \mathbb{R}^k$ is onto.

Canonical Submersion Thoerem. *Assume that f is a submersion at p and that $f(p) = 0$. Then there exists a neighborhood U_0 of p in U, a neighborhood V of 0 in $\mathbb{R}^n$, and a diffeomorphism $g : V \to U_0$ such that $f \circ g = \pi$.*

Proof. Let $T_p : \mathbb{R}^n \to \mathbb{R}^n$ be the translation defined by $x \to x + p$. Replacing f by $f \circ T_p$ we can assume that $p = 0$ and $f(0) = 0$.

Let $A = (Df)(0)$, where $A : \mathbb{R}^n \to \mathbb{R}^k$ is onto by the assumption that f is a submersion. So, there exists a bijective linear map $B : \mathbb{R}^n \to \mathbb{R}^n$ such that $A \circ B = \pi$. Replacing f by $f \circ B$ we can assume that $Df(0) = \pi$.

Define a map $h : U \to \mathbb{R}^n$ by

$$h(x_1, \dots, x_n) = (f(x_1, \dots, x_k); x_{k+1}, \dots, x_n). \tag{6.3}$$

Note that (1) $Dh(0) = I$; and (2) $\pi h = f$. By (1), the function h maps a neighborhood U_0 of 0 in U diffeomorphically onto a neighborhood V of 0 in $\mathbb{R}^n$. By (2), we have $\pi = f \circ h^{-1}$. Take $g = h^{-1}$. □

There is a companion theorem having to do with immersions.

Definition 6.3. Let U be an open subset of $\mathbb{R}^k$, and let $f : U \to \mathbb{R}^n$ be a $\mathcal{C}^\infty$ map. Let $p \in U$. The map f is an *immersion at p* if $(Df)(p) : \mathbb{R}^k \to \mathbb{R}^n$ is injective (one-to-one).

Canonical Immersion Thoerem. *Let U be a neighborhood of 0 in $\mathbb{R}^k$, and let $f : U \to \mathbb{R}^n$ be a $\mathcal{C}^\infty$ map. Assume that f is an immersion at 0. Then there exists a neighborhood V of $f(0) = p$ in $\mathbb{R}^n$, a neighborhood W of 0 in $\mathbb{R}^k$, and a diffeomorphism $g : V \to W$ such that $\iota^{-1}(W) \subseteq U$ and $g \circ f = \iota$.*

Proof. Replacing f by $T_p \circ f$, we can assume that $f(0) = 0$. Let $A = Df(0)$, so $A : \mathbb{R}^k \to \mathbb{R}^n$ is injective. There exists a linear map $B : \mathbb{R}^n \to \mathbb{R}^n$ such that $BA = \iota$. Replacing f by $B \circ f$, we can assume that $Df(0) = \iota$.

Let $\ell = n - k$. Since $U \subseteq \mathbb{R}^k$, we get $U \times \mathbb{R}^\ell \subseteq \mathbb{R}^k \times \mathbb{R}^\ell = \mathbb{R}^n$. Define a map $h : U \times \mathbb{R}^\ell \to \mathbb{R}^n$ by

$$h(x_1, \dots, x_n) = f(x_1, \dots, x_k) + (0, \dots, 0, x_{k+1}, \dots, x_n). \tag{6.4}$$

One can check that (1) $Dh(0) = I$; and (2) $h \circ \iota = f$.

By (1), the function h maps a neighborhood W of 0 in $U \times \mathbb{R}^\ell$ diffeomorphically onto a neighborhood V of 0 in $\mathbb{R}^n$. Moreover, $W \subseteq U \times \mathbb{R}^\ell$, so $\iota^{-1}(W) \subseteq U$.

By (2), we obtain the canonical immersion map $\iota = h^{-1} \circ f$. Take $g = h^{-1}$. □

6.2 Definition of Manifold

Now we move on to the study of manifolds.

Let X be a subset of $\mathbb{R}^n$, let Y be a subset of $\mathbb{R}^m$, and let $f : X \to Y$ be a continuous map. We define that the map f is a $\mathcal{C}^\infty$ map if for every point $p \in X$ there exists a neighborhood U_p of p in $\mathbb{R}^n$ and a $\mathcal{C}^\infty$ map $g_p : U_p \to \mathbb{R}^n$ such that $g_p | X \cap U_p = f$.

We showed in the homework that if $f : X \to Y$ is a $\mathcal{C}^\infty$ map, then there exists a neighborhood U of X in $\mathbb{R}^n$ and a $\mathcal{C}^\infty$ map $g : U \to \mathbb{R}^n$ extending f.

Definition 6.4. A map $f : X \to Y$ is a *diffeomorphism* if it is one-to-one, onto, a $\mathcal{C}^\infty$ map, and $f^{-1} : Y \to X$ is $\mathcal{C}^\infty$.

Let X be a subset of $\mathbb{R}^N$.

Definition 6.5. The set X is an *n-dimensional manifold* if for every point $p \in X$ there exists a neighborhood V of p in $\mathbb{R}^N$, an open set U in $\mathbb{R}^m$, and a diffeomorphism $f : U \to V \cap X$. The collection (f, U, X) is called a *parameterization of X at p*.

This definition does not illustrate how manifolds come up in nature. Usually manifolds come up in the following scenario.

Let W be open in $\mathbb{R}^N$, and let $f_i : W \to \mathbb{R}$, $i = 1, \dots, \ell$ be $\mathcal{C}^\infty$ functions. Suppose you want to study the solution space of

$$f_i(x_1, \dots, x_N) = 0, \quad i = 1, \dots, \ell. \tag{6.5}$$

Then you consider the mapping $f : W \to \mathbb{R}^\ell$ defined by

$$f(x) = (f_1(x), \dots, f_\ell(x)). \tag{6.6}$$

Claim. *If for every $p \in W$ the map f is a submersion of p, then Equation 6.6 defines a k-dimensional manifold, where $k = N - \ell$.*