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18.102 Introduction to Functional Analysis
Spring 2009

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PROBLEM SET 9 FOR 18.102, SPRING 2009
DUE 11AM TUESDAY 28 APR.

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Corrections to earlier versions

My apologies for all these errors. Here is a list – they are fixed below (I hope).

(1) In P9.2 (2), and elsewhere, $\mathcal{C}^\infty(\mathbb{S})$ should be $\mathcal{C}^0(\mathbb{S})$, the space of continuous functions on the circle – with supremum norm.
(2) In (P9.2.9) it should be $u = Fv$, not $u = Sv$.
(3) Similarly, before (P9.2.10) it should be $u = Fv$.
(4) Discussion around (P9.2.12) clarified.
(5) Last part of P10.2 clarified.

This week I want you to go through the invertibility theory for the operator

$$Qu = (-\frac{d^2}{dx^2} + V(x))u(x) \tag{P9.1}$$

acting on periodic functions. Since we have not developed the theory to handle this directly we need to approach it through integral operators.

Before beginning, we need to consider periodic functions.

1. P9.1: Periodic functions

Let $\mathbb{S}$ be the circle of radius 1 in the complex plane, centered at the origin, $\mathbb{S} = \{z; |z| = 1\}$.

(1) Show that there is a 1-1 correspondence

$$\begin{aligned}\mathcal{C}^0(\mathbb{S}) = \{u : \mathbb{S} \longrightarrow \mathbb{C},\ \text{continuous}\} \longrightarrow \\ \{u : \mathbb{R} \longrightarrow \mathbb{C};\ \text{continuous and satisfying } u(x+2\pi) = u(x)\ \forall\ x \in \mathbb{R}\}.\end{aligned} \tag{P9.1.1}$$

(2) Show that there is a 1-1 correspondence

$$\begin{aligned}L^2(0,2\pi) \longleftrightarrow \{u \in \mathcal{L}^1_{\mathrm{loc}}(\mathbb{R}); u\big|_{(0,2\pi)} \in \mathcal{L}^2(0,2\pi) \\ \text{and } u(x+2\pi) = u(x)\ \forall\ x \in \mathbb{R}\}/\mathcal{N}_P\end{aligned} \tag{P9.1.2}$$

where $\mathcal{N}_P$ is the space of null functions on $\mathbb{R}$ satisfying $u(x+2\pi) = u(x)$ for all $x \in \mathbb{R}$.

(3) If we denote by $L^2(\mathbb{S})$ the space on the left in (P9.1.2) show that there is a dense inclusion

$$\mathcal{C}^0(\mathbb{S}) \longrightarrow L^2(\mathbb{S}). \tag{P9.1.3}$$

So, the idea is that we can think of functions on $\mathbb{S}$ as 2π-periodic functions on $\mathbb{R}$.

2. P9.2: Schrödinger's operator

Since that is what it is, or at least it is an example thereof:

$$-\frac{d^2u(x)}{dx^2}+V(x)u(x)=f(x),\ x\in\mathbb{R}, \tag{P9.2.1}$$

(1) First we will consider the special case $V=1$. Why not $V=0$? – Don't try to answer this until the end!
(2) Recall how to solve the differential equation

$$-\frac{d^2u(x)}{dx^2}+u(x)=f(x),\ x\in\mathbb{R}, \tag{P9.2.2}$$

where $f(x)\in\mathcal{C}^0(\mathbb{S})$ is a continuous, 2π-periodic function on the line. Show that there is a unique 2π-periodic and twice continuously differentiable function, u, on $\mathbb{R}$ satisfying (P9.2.2) and that this solution can be written in the form

$$u(x)=(Sf)(x)=\int_{0,2\pi}A(x,y)f(y) \tag{P9.2.3}$$

where $A(x,y)\in\mathcal{C}^0(\mathbb{R}^2)$ satisfies $A(x+2\pi,y+2\pi)=A(x,y)$ for all $(x,y)\in\mathbb{R}$.

Extended hint: In case you managed to avoid a course on differential equations! First try to find a solution, igonoring the periodicity issue. To do so one can (for example, there are other ways) factorize the differential operator involved, checking that

$$-\frac{d^2u(x)}{dx^2}+u(x)=-(\frac{dv}{dx}+v)\text{ if }v=\frac{du}{dx}-u \tag{P9.2.4}$$

since the cross terms cancel. Then recall the idea of integrating factors to see that

$$\begin{aligned}\frac{du}{dx}-u&=e^x\frac{d\phi}{dx},\ \phi=e^{-x}u,\\ \frac{dv}{dx}+v&=e^{-x}\frac{d\psi}{dx},\ \psi=e^xv.\end{aligned} \tag{P9.2.5}$$

Now, solve the problem by integrating twice from the origin (say) and hence get a solution to the differential equation (P9.2.2). Write this out explicitly as a double integral, and then change the order of integration to write the solution as

$$u'(x)=\int_{0,2\pi}A'(x,y)f(y)dy \tag{P9.2.6}$$

where A' is continuous on $\mathbb{R}\times[0,2\pi]$. Compute the difference $u'(2\pi)-u'(0)$ and $\frac{du'}{dx}(2\pi)-\frac{du'}{dx}(0)$ as integrals involving f. Now, add to u' as solution to the *homogeneous* equation, for $f=0$, namely $c_1e^x+c_2e^{-x}$, so that the new solution to (P9.2.2) satisfies $u(2\pi)=u(0)$ and $\frac{du}{dx}(2\pi)=\frac{du}{dx}(0)$. Now, check that u is given by an integral of the form (P9.2.3) with A as stated.
(3) Check, either directly or indirectly, that $A(y,x)=A(x,y)$ and that A is real.
(4) Conclude that the operator S extends by continuity to a bounded operator on $L^2(\mathbb{S})$.

(5) Check, probably indirectly rather than directly, that

$$S(e^{ikx}) = (k^2+1)^{-1}e^{ikx},\ k \in \mathbb{Z}. \tag{P9.2.7}$$

(6) Conclude, either from the previous result or otherwise that S is a compact self-adjoint operator on $L^2(\mathbb{S})$.

(7) Show that if $g \in \mathcal{C}^0(\mathbb{S}))$ then Sg is twice continuously differentiable. Hint: Proceed directly by differentiating the integral.

(8) From (P9.2.7) conclude that $S = F^2$ where F is also a compact self-adjoint operator on $L^2(\mathbb{S})$ with eigenvalues $(k^2+1)^{-\frac{1}{2}}$.

(9) Show that $F : L^2(\mathbb{S}) \longrightarrow \mathcal{C}^0(\mathbb{S})$.

(10) Now, going back to the real equation (P9.2.1), we assume that V is continuous, real-valued and 2π-periodic. Show that if u is a twice-differentiable 2π-periodic function satisfying (P9.2.1) for a given $f \in \mathcal{C}^0(\mathbb{S})$ then

$$u + S((V-1)u) = Sf \text{ and hence } u = -F^2((V-1)u) + F^2 f \tag{P9.2.8}$$

and hence conclude that

$$u = Fv \text{ where } v \in L^2(\mathbb{S}) \text{ satisfies } v + (F(V-1)F)v = Ff \tag{P9.2.9}$$

where $V-1$ is the operator defined by multiplication by $V-1$.

(11) Show the converse, that if $v \in L^2(\mathbb{S})$ satisfies

$$v + (F(V-1)F)v = Ff,\ f \in \mathcal{C}^0(\mathbb{S}) \tag{P9.2.10}$$

then $u = Fv$ is 2π-periodic and twice-differentiable on $\mathbb{R}$ and satisfies (P9.2.1).

(12) Apply the Spectral theorem to $F(V-1)F$ (including why it applies) and show that there is a sequence λ_j in $\mathbb{R} \setminus \{0\}$ with $|\lambda_j| \to 0$ such that for all $\lambda \in \mathbb{C} \setminus \{0\}$, the equation

$$\lambda v + (F(V-1)F)v = g,\ g \in L^2(\mathbb{S}) \tag{P9.2.11}$$

has a unique solution for every $g \in L^2(\mathbb{S})$ if and only if $\lambda \neq \lambda_j$ for any j.

(13) Show that for the λ_j the solutions of

$$\lambda_j v + (F(V-1)F)v = 0, v \in L^2(\mathbb{S}), \tag{P9.2.12}$$

are all continuous 2π-periodic functions on $\mathbb{R}$.

(14) Show that the corresponding functions $u = Fv$ where v satisfies (P9.2.12) are all twice continuously differentiable, 2π-periodic functions on $\mathbb{R}$ satisfying

$$-\frac{d^2u}{dx^2} + (1 - s_j + s_j V(x))u(x) = 0,\ s_j = 1/\lambda_j. \tag{P9.2.13}$$

(15) Conversely, show that if u is a twice continuously differentiable, 2π-periodic function satisfying

$$-\frac{d^2u}{dx^2} + (1 - s + sV(x))u(x) = 0,\ s \in \mathbb{C}, \tag{P9.2.14}$$

and u is not identically 0 then $s = s_j$ for some j.

(16) Finally, conclude that Fredholm's alternative holds for the equation (P9.2.1)

Theorem 1. *For a given real-valued, continuous 2π-periodic function V on $\mathbb{R}$, either* (P9.2.1) *has a unique twice continuously differentiable, 2π-periodic, solution for each f which is continuous and 2π-periodic or else* *there exists a finite, but positive, dimensional space of twice continuously differentiable 2π-periodic solutions to the homogeneous equation*

$$-\frac{d^2w(x)}{dx^2} + V(x)w(x) = 0,\ x \in \mathbb{R}, \tag{P9.2.15}$$

and (P9.2.1) *has a solution if and only if* $\int_{(0,2\pi)} fw = 0$ *for every* 2π*-periodic solution,* w*, to* (P9.2.15).

3. Not to be handed in, just for the enthusiastic

Check that we really can understand all the 2π periodic eigenfunctions of the Schrödinger operator using the discussion above. First of all, there was nothing sacred about the addition of 1 to $-d^2/dx^2$, we could add any positive number and get a similar result – the problem with 0 is that the constants satisfy the homogeneous equation $d^2u/dx^2 = 0$. What we have shown is that the operator

$$u \longmapsto Qu = -\frac{d^2u}{dx^2}u + Vu \tag{P9.2.1}$$

applied to twice continuously differentiable functions has at least a left inverse unless there is a non-trivial solution of

$$-\frac{d^2u}{dx^2}u + Vu = 0. \tag{P9.2.2}$$

Namely, the left inverse is $R = F(\mathrm{Id} + F(V-1)F)^{-1}F$. This is a compact self-adjoint operator. Show – and there is still a bit of work to do – that (twice continuously differentiable) eigenfunctions of Q, meaning solutions of $Qu = \tau u$ are precisely the non-trivial solutions of $Ru = \tau^{-1}u$.

What to do in case (P9.2.2) does have a non-trivial solution? Show that the space of these is finite dimensional and conclude that essentially the same result holds by working on the orthocomplement in $L^2(\mathbb{S})$.

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