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18.102 Introduction to Functional Analysis  
Spring 2009

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**PROBLEM SET 8 FOR 18.102, SPRING 2009**  
**DUE 11AM TUESDAY 14 APRIL.**

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Okay, I forgot to put the problems up. So, here are three problems that should be reasonably quick. If anyone is seriously inconvenienced by the limited time they have to work on them, just let me know and I will give you a couple of days.

1. PROBLEM 8.1

Show that a continuous function  $K : [0, 1] \rightarrow L^2(0, 2\pi)$  has the property that the Fourier series of  $K(x) \in L^2(0, 2\pi)$ , for  $x \in [0, 1]$ , converges uniformly in the sense that if  $K_n(x)$  is the sum of the Fourier series over  $|k| \leq n$  then  $K_n : [0, 1] \rightarrow L^2(0, 2\pi)$  is also continuous and

$$(1) \quad \sup_{x \in [0, 1]} \|K(x) - K_n(x)\|_{L^2(0, 2\pi)} \rightarrow 0.$$

Hint. Use one of the properties of compactness in a Hilbert space that you proved earlier.

2. PROBLEM 8.2

Consider an integral operator acting on  $L^2(0, 1)$  with a kernel which is continuous –  $K \in \mathcal{C}([0, 1]^2)$ . Thus, the operator is

$$(1) \quad Tu(x) = \int_{(0, 1)} K(x, y)u(y).$$

Show that  $T$  is bounded on  $L^2$  (I think we did this before) and that it is in the norm closure of the finite rank operators.

Hint. Use the previous problem! Show that a continuous function such as  $K$  in this Problem defines a continuous map  $[0, 1] \ni x \mapsto K(x, \cdot) \in \mathcal{C}([0, 1])$  and hence a continuous function  $K : [0, 1] \rightarrow L^2(0, 1)$  then apply the previous problem with the interval rescaled.

Here is an even more expanded version of the hint: You can think of  $K(x, y)$  as a continuous function of  $x$  with values in  $L^2(0, 1)$ . Let  $K_n(x, y)$  be the continuous function of  $x$  and  $y$  given by the previous problem, by truncating the Fourier series (in  $y$ ) at some point  $n$ . Check that this defines a finite rank operator on  $L^2(0, 1)$  – yes it maps into continuous functions but that is fine, they are Lebesgue square integrable. Now, the idea is the difference  $K - K_n$  defines a bounded operator with small norm as  $n$  becomes large. It might actually be clearer to do this the other way round, exchanging the roles of  $x$  and  $y$ .

## 3. PROBLEM 8.3

Although we have concentrated on the Lebesgue integral in one variable, you proved at some point the covering lemma in dimension 2 and that is pretty much all that was needed to extend the discussion to 2 dimensions. Let's just assume you have assiduously checked everything and so you know that  $L^2((0, 2\pi)^2)$  is a Hilbert space. Sketch a proof – noting anything that you are not sure of – that the functions  $\exp(ikx + ily)/2\pi$ ,  $k, l \in \mathbb{Z}$ , form a complete orthonormal basis.

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