18.102 Introduction to Functional Analysis
Spring 2009

SOLUTIONS TO PROBLEMS 6

Hint: Don't pay too much attention to my hints, sometimes they are a little off-the-cuff and may not be very helpfult. An example being the old hint for Problem 6.2!

Problem 6.1 Let H be a separable Hilbert space. Show that $K \subset H$ is compact if and only if it is closed, bounded and has the property that any sequence in K which is weakly convergent sequence in H is (strongly) convergent.

Hint:- In one direction use the result from class that any bounded sequence has a weakly convergent subsequence.

Problem 6.2 Show that, in a separable Hilbert space, a weakly convergent sequence $\{v_n\}$, is (strongly) convergent if and only if the weak limit, v satisfies

$$\|v\|_H = \lim_{n\to\infty} \|v_n\|_H. \tag{14.28}$$

Hint:- To show that this condition is sufficient, expand

$$(v_n - v, v_n - v) = \|v_n\|^2 - 2\operatorname{Re}(v_n, v) + \|v\|^2. \tag{14.29}$$

Problem 6.3 Show that a subset of a separable Hilbert space is compact if and only if it is closed and bounded and has the property of 'finite dimensional approximation' meaning that for any $\epsilon > 0$ there exists a linear subspace $D_N \subset H$ of finite dimension such that

$$d(K, D_N) = \sup_{u\in K} \inf_{v\in D_N} \{d(u,v)\} \le \epsilon. \tag{14.30}$$

Hint:- To prove necessity of this condition use the 'equi-small tails' property of compact sets with respect to an orthonormal basis. To use the finite dimensional approximation condition to show that any weakly convergent sequence in K is strongly convergent, use the convexity result from class to define the sequence $\{v'_n\}$ in D_N where v'_n is the closest point in D_N to v_n. Show that v'_n is weakly, hence strongly, convergent and hence deduce that $\{v_n\}$ is Cauchy.

Problem 6.4 Suppose that $A : H \longrightarrow H$ is a bounded linear operator with the property that $A(H) \subset H$ is finite dimensional. Show that if v_n is weakly convergent in H then Av_n is strongly convergent in H.

Problem 6.5 Suppose that H_1 and H_2 are two different Hilbert spaces and $A : H_1 \longrightarrow H_2$ is a bounded linear operator. Show that there is a unique bounded linear operator (the adjoint) $A^* : H_2 \longrightarrow H_1$ with the property

$$\langle Au_1, u_2\rangle_{H_2} = \langle u_1, A^*u_2\rangle_{H_1} \ \forall\ u_1 \in H_1,\ u_2 \in H_2. \tag{14.31}$$