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CASEY RODRIGUEZ: We defined the Lebesgue integral for simple functions, which have this canonical representation as a finite linear combination of indicator functions on sets which are pairwise disjoint and whose union gives me E. We defined their integral-- the integral of phi was defined to be the sum from j equals 1 to n aj times the measure of capital A, sum as j goes from 1 to n.

And we proved a couple of properties of this. Namely, we proved that if I multiply a simple function by a non-negative scalar, then the scalar pulls out of the integral. The integral of the scalar multiple of phi is equal to scalar multiple of the integral. If I have two non-negative simple functions and I add them, I get another non-negative simple function and the integral of the sum is the sum of the integrals.

And we also prove that if I have two simple functions, one less than or equal to the other, then the integrals respect this inequality. The integral of the smaller one is less than or equal to the integral of the bigger one. So now, we're going to define the Lebesgue integral for a general non-negative measurable function.

In some sense, how one should view the Lebesgue is kind of how one should view the Riemann integral, I guess, if you-- or at least when you think about the Riemann integral as you build up these approximations to the integral by cutting up the domain for the Riemann integral. And you choose points in between and you form these boxes.

Now, you have some freedom in how you choose these boxes that approximate the integral of f. But one way to choose the boxes-- or at least when I picture it, I always picture the boxes sitting below the graph of f. And as you dice up the domain smaller and smaller then these things are kind of-- these boxes are getting narrower and kind of your approximation is filling in the integral or the area underneath the curve from below.

And we've already seen that for non-negative measurable functions. There always exists a sequence of simple functions that increase to f. So that for every x I stick into the sequence of functions-- phi 1 is less than or equal to phi 2 is less than or equal to phi 3 and so on-- and these phis are increasing pointwise to f.

So if you're trying to build this on your own, you would think, OK, let me define the integral as of a function, of a non-negative measurable function, as the limit of the integrals of a sequence of simple functions increasing to f. Which I know exists, because we constructed one. Of course-- and that's one way to do it, and some textbooks do that.

But you come across this problem that this number that you've defined to be the limit of these integrals-- maybe it depends on the sequence of measurable functions that you took in the beginning. So we're not going to quite do that. In the end, we'll see that this number we defined can be given as of the limit of the integrals of simple functions.

So for a general non-negative measurable functions, we define the integral of f over E to be the sup of the integral over E of phi, where now, phi is a non-negative simple function and phi sits below f. So in some sense, this is kind of like taking the integral of f to be defined as kind of-- I don't want to say limit, because it's not exactly a limit, but in some sense, the thing that's getting filled up by all the integrals of the simple functions below the graph that sit below the graph of f.

OK, so let me just prove a very simple theorem, which is useful. So if E is a subset of R is measurable-- in fact, it's a set of measures 0. So remember, all sets of outer measure 0 are measurable.

So I don't have to say really if E is a subset of R with E measurable. When I say this, I'm kind of saying two things at once. It's outer measure of E is 0 or 1, and therefore it's measurable. So then for all f that are non-negative and measurable on E, the integral over E of f is 0.

OK, so it's only kind of interesting to take the integral over sets that have positive measure. So no matter what function I take, the integral over a set of measure 0 is 0. OK, this is kind of akin to, in Riemann integration, the integral over a point being 0. But now we have more interesting sets of measure 0 other than just a point.

So what's the proof of this? Well, we don't have much to go off of. We just have the definition. So let's use the definition. Let phi be simple with this sort of canonical representation with the less than or equal to f.

So what I'd like to show is that the integral of phi is 0. But this is-- and therefore, the integral of f, which is the sup over all of these integrals, is therefore 0. But this is clear because since all of these Aj's are subsets of E, which is a set of measure 0, this implies the measure of Aj equals 0 for all j. And therefore, the integral of phi over E, which is equal to Aj measure of Aj equals 0.

And thus, the integral of f, which is the sup over all of these, is just a sup of 0 then. So I guess I should have started this proof off with let f be an L plus of E. So I left that off, but you know what I was doing, hopefully.

OK, so it's only interesting to take the integral over sets of positive measure. Now, we have a few facts which carry over. Well, not really carry. Well, one of them carries over from what we did for simple functions and the others just kind of follow from the definition.

And it'll be an exercise in the assignment, so it's the following. If phi is-- then the two definitions of the integral of phi agree. And so I'll say exactly what I mean in just a second.

Two, f and g are in L plus of E. c is a non-negative real number and f is less than or equal to g on E. Then a couple of things. The integral of c times f is equal to c times the integral of f and the integral over E of f is less than or equal to the integral of g over E.

And one final property is the following. If f is a measurable function on E and f is a measurable subset of E, then I can integrate f also on capital F. So this is in the, I think, maybe assignment 5 that if I have a function which is measurable on a set and I take any subset of that measurable set, which is measurable, then f restricted to this set is also measurable.

So what I'm about to say makes sense. Then the integral of little f over capital F is equal to the integral of E of little f times the indicator function of f, which is less than or equal to the integral over E of little f.

So I think the rest of the statements, statements 2 and 3 are completely unambiguous. Maybe you were wondering what exactly did I mean by 1. So we had kind of two definitions of the integral of a simple function, right?

We defined it first as the sum of the coefficients times the measure of these sets. And then we also have a second definition because a simple function-- a measurable non-negative simple function is also in this set. So I should have that what's here should be equal to what's on the right.

So statement 1 is the statement that what's underlined three times is equal to what was underlined four times. That's a lot of lines. All right, so this will be a fairly straightforward exercise just using the definitions. And also, what we did for simple functions.

One consequence of this theorem and the one before it is the following. Is that I can relax this condition here in 2 to an almost everywhere statement. So if f and g are non-negative measurable functions, and f is less than or equal to g almost everywhere on E, then the integral of E of f is less than or equal to the integral of g.

So let's write the proof of this. So let f be the set of all x's and E such that f of x is less than or equal to g of x. So it's not difficult to realize that this is a measurable set. This is, if you'd like, f minus g.

The inverse image of-- or let's do it this way. Write this as g minus f inverse image of 0 infinity. And since g and f are measurable functions, their difference is measurable, non-negative. So this is always non-negative.

OK, so maybe there's a small issue with what happens at infinity, but you're dealing with that in the assignment. So I'm just going to erase this from the board and you'll just have to accept that this is measurable under the wisdom that I gave you that if you can write it down, typically it's measurable. And what is the measure of the complement is 0 because this is supposed to hold almost everywhere.

OK, so I left off the fact, which strictly speaking, we might need for this, but. OK, so this is also an exercise. So then the integral of little f over E, which is equal to the integral over E of f union, f complement here-- this is equal to the integral over f, little f, plus the integral over f complement of little f

So strictly speaking I didn't write why this is true down. But let's think about it just for a moment. So these are two disjoint subsets that make up E. Why is this going to be equal to this? Well, it's true for simple functions.

So I mean, if I make this statement and assume phi-- I mean, f is simple, then it's not hard to convince yourself that this is true. And if it's true for simple functions, then from how we've defined the integral as to be this sup, it will carry over to general, non-negative measurable functions. So let's just accept this and you can prove it on your own. It's not difficult.

But since we have this and f complement is a set of measure 0, this is the integral of little f. All right, since this is 0. Now, on capital F, little f is less than or equal to g. So this is less than or equal to the integral of g over capital F.

And just going backwards, this is equal to the integral of fg plus f complement, which is equal to the integral of g, G. So modulo this equality, which I leave to you to fill in, proves the theorem. OK, so now we have the definition of Lebesgue integral for non-negative measurable functions. We have some properties of it.

What's kind of missing from this list that I've given so far is that linearity, I guess, right? The integral of the sum of two non-negative measurable functions is equal to the sum of the integrals. We had that for simple functions. How do we get that for general non-negative measurable functions?

OK, so what I'm about to prove is not just a tool for proving that but is one of the big three convergence theorems that you find in Lebesgue measure in integration or Lebesgue integration, which is the following-- monotone convergence theorem, which is the following that if fn is a sequence of non-negative measurable functions such that f1 is less than or equal to f2 is less than or equal to f3 on E.

So pointwise they are increasing. The sequence is increasing. And there exists a function f so that fn goes to f pointwise on E. So let me just recall, this means for all x and E limit as n goes to infinity of fn of x equals f of x.

So in particular, f is going to be a non-negative measurable function because remember, the pointwise limit of measurable functions is measurable, so what I'm about to say makes sense. Then the integrals converge to the integrals of the limit. So the limit of the integrals is the integral of the limits.

So this is a much stronger statement than anything you come across in Riemann integration. Riemann integration usually requires uniform convergence while here, at least for monotone sequences, we just need pointwise convergence. So I think there is a version of this theorem that one could state for Riemann integration.

But still, just on the face of it, you have a pointwise statement implying convergence of integrals. So that should immediately kind of suggest to you that what we've built up, this Lebesgue integration, is much more powerful than Riemann integration. So let's prove the theorem.

So since f1 is less than or equal to f2 is less than or equal to f3 and so on, this implies the integral of E of f1 is less than or equal to the integral of f2 and so on. And what else? So-- which implies that the limit as n goes to infinity of the integrals of fn exists in 0 infinity.

So this is a non-negative increasing sequence of real numbers now. The integral over E of f1, that's a real number. This is a real number.

All of these are non-negative numbers because this is a sup over non-negative numbers. So I have an increasing sequence of non-negative numbers. So that either has a limit, a finite limit, or it must go to infinity. That's not difficult to prove knowing what from basic analysis.

Moreover, since their pointwise increasing and converging for all x, this implies that we still have f1 is less than or equal to f3 and so on. But they all sit below f. So for each x-- this is a f1 of x, f2 of x, f3 of x and so on this is an increasing sequence of real numbers converging to f of x, which is either a finite number or infinity. . So these numbers are increasing to this limit, and therefore, they must always sit below the limit.

And since all of these functions sit below f, this implies that for all n, the integral of fn over E is less than or equal to the integral of f. And therefore, the limit, which we know exists as either a finite number or infinity, is less than or equal to the integral of f. So just based on the assumptions, we immediately get that one of these quantities that we want to show is equal to the other is less than or equal to the other quantity.

So standard trick of analysis. If you get kind of for free one quantity is less than or equal to the other quantity that you want to show are equal, let's try and go the reverse direction. So now we show that the integral of f over E is less than or equal to the integral of the limit as n goes to infinity of the integral over E of fn. And therefore, the two are equal.

All right. Now to show this, we're going to show that for every simple function less than or equal to f, the integral of that simple function sits below this. Now, we know these fn's so here's the game plan. We know the fn's are increasing to f.

So if I take a simple function less than or equal to f-- if the simple function is less than f, then eventually, fn is going to pass it up, right? Because the fn's are increasing to f and the simple function sits below f. And therefore, eventually, we should have this is bigger than or equal to the integral of that simple function.

Now, we're only requiring the simple function to be less than or equal to f so we'll give ourselves a little bit of room and then send that bit of room to 0. So let phi be a non-negative simple function. Phi equals sum j equals 1 to n Aj chi Aj with phi less than or equal to f. And our goal is to show that the integral of phi is less than or equal to the limit as n goes to infinity of fn.

So here's that little bit of room I was referring to. Let epsilon be a small number between 0 and 1. And let En be the set of all x's and E such that fn of x is greater than or equal to 1 minus epsilon f of x-- I mean, phi of x. So note for all x in E, 1 minus epsilon times phi of x, this is strictly less than now f of x.

So we had phi of x is less than or equal to f of x. But now, if I multiply this by a small number that's close to-- or a number that's slightly less than 1, then I will have strict inequality. So in particular, then since for all x and E I have a-- equals f of x. This implies that every x must eventually lie in one of these E sub fn's right? Because 1 minus epsilon phi of x is less than f of x.

The fn's are approaching f of x so it must pass up this value at some point in its quest to get to f of x, or at least close to f of x. So simply from this fact, this implies that the union over n equals 1 to infinity of the En's gives me E. Let me highlight this.

Now, since these functions are increasing-- I should say, they're pointwise increasing. Not that they are increasing functions, but they are pointwise increasing. So f1 is less than or equal to f2 is less than or equal to f3 and so on. This implies that E1 is contained in E2 is contained in E3 and so on.

If I have some n so that x is in this set-- so fn of x is bigger than or equal to 1 minus epsilon phi of x-- then f of n plus 1 of x is bigger than or equal to fn of x, which is bigger than or equal to 1 minus epsilon phi of x. And therefore, that x is then En plus 1. So this is not only a sequence of sets whose union gives me E, they're an increasing sequence of sets. Increasing in the sense of inclusion. OK?

Now, we're going to use these two highlighted things in just a minute, along with continuity of Lebesgue measure to get what we want. So we have the integral of E fn. This is less than or equal to the integral of fn over a smaller set, so E sub n. Now, on E sub n-- remember, the E sub n's are defined as where fn is bigger than or equal to 1 minus epsilon phi of x.

So this is bigger than or equal to E sub n 1 minus phi-- sorry, what am I doing? Minus epsilon phi, which is equal to 1 minus epsilon times the integral of E sub n over phi. And this is, by definition, equal to a sum from j equals 1 to n. Over the set E sub n, I get integral Aj measure of Aj intersect E sub n.

And I made a small notational error. Let's change this into an m since we have n already denoting the-- indexing the functions, we do not want this n right here. So that should be an m. It's just a fixed finite number depending on the simple function.

So I have this for all n. And therefore, the limit as n goes to infinity of the integral over E of f sub n is less than or equal to the limit as n goes to infinity of 1 minus epsilon times sum from j equals 1 to m Aj measure of Aj E sub n. Now, the E sub n's are increasing to E, and therefore, for each fixed j-- so in fact, let's pause on this real quick and come back to this thing.

We're going to eventually take the limit as n goes to infinity of this quantity. So let's look at what this does as n goes to infinity since by those two things that I highlighted, that En's are increasing subsets of E whose union gives me E, I get-- since E1 Aj is a subset of E2 Aj subset of E3 and so on. And the union n equals 1 to infinity of E sub n intersect Aj equals Aj, because then this is just going to be equal to the union of the E sub n's intersect Aj. That just gives me E intersect Aj, which is just Aj.

We get by the continuity of Lebesgue measure-- this implies for all j-- that the limit as n goes to infinity of the measure of Aj intersect En is equal to the measure of Aj, which, as we just said, this En intersect Aj, which-- remember, this set is equal to Aj, so this is measure of Aj. So from the two yellow boxes we had before, we get this useful one.

For all j, we have the limit as n goes to infinity of the measure of Aj intersect En is equal to the measure of Aj. So now, we'll stick this into this inequality after we take the limit. So I got ahead of myself a minute ago.

Thus, limit as n goes to infinity of the integral over E of fn, this is bigger than or equal to limit as n goes to infinity of 1 minus epsilon m now Aj measure of Aj intersect En. Now, these numbers here all converge to-- each of these numbers here for each j converges as n goes to infinity to the measure of A sub j. So the limit is then equal to 1 minus epsilon times the integral j equals 1 to m times-- what am I writing the integral for?

Aj measure of Aj. And this is just equal to 1 minus integral, by definition, the integral of phi. So I've shown that for all epsilon between 0 and 1, this 1 minus epsilon times the integral of phi is less than or equal to this number over here, which may be infinite, may be finite. And since this hold for all epsilon, I can send epsilon to 0.

So I have this inequality between now fixed things along with an epsilon here, so I can send epsilon to 0 and I get the integral of phi is less than or equal to the limit as n goes to infinity of fn. And since phi is an arbitrary simple function that's less than or equal to f, the sup over all of these-- which is, by definition, the integral of f-- is less than or equal to the limit of the integrals.

All right. That's the end of the proof. So monotone convergence theorem, a very useful theorem, important theorem throughout all of this. OK, so let's get a few applications from this. So this first one is kind of a way how would you evaluate now this integral? Remember, the integral, which I just erased, is defined as the sup over all integrals of simple functions.

So in order to actually compute the integral of a non-negative measurable function, I would have to find out the integral of every simple function less than or equal to it and take the sup over that whole set, which is kind of a useless or impossible way of computing the integral. It's similar to when you come across Riemann integration and the Riemann integral is defined as the limit of Riemann sums. Nobody-- you can compute maybe three integrals just from Riemann sum.

So we need a more efficient way of being able to compute the Lebesgue integral, and the monotone convergence theorem gives us that kind of for free. So we have the following, if f is a non-negative measurable function and phi n is a sequence of simple functions, which are all non-negative and pointwise increasing and converging pointwise to f then the integral over E of f is equal to the limit as n goes to infinity of the integral of the simple functions.

So back when we discussed measurable functions, we actually constructed such a sequence of simple functions that satisfies the hypotheses of this theorem. So this is not a vacuous theorem. But this theorem tells you that if you want to compute the integral of f, just take any sequence of simple functions increasing up to f and compute the limit of the integral. And that'll give you the integral of f.

Now, there's just this-- this follows immediately from the monotone convergence theorem. I have the taking fn is equal to the phi n's. So there's no proof to go with that.

The next theorem is linearity of the integral. So if f and g are two non-negative measurable functions, then the integral of f plus g is equal to the integral of f plus the integral of g. Now, note there's no ambiguity with how to define, so we kind of had a-- there's some touchy business about adding and subtracting two extended real valued measurable functions, but there's none of that here since these are both non-negative measurable extended real valued functions. So this will always only be of the form infinity plus infinity, which we define to be infinity. So just let me make that small note.

So the integral is linear, so what's the proof? Let phi n and psi n be two sequences of simple functions such that they're increasing to f and g, respectively. So 0 is less than or equal to phi 1 is less than or equal to phi 2 and so on. And phi n converges to f pointwise on E.

OK, so I should have-- everything's happening on this set E. And the same for the psis. And psi n converging to g pointwise.

Then, if I take the sum of these two simple functions or sequences of simple functions, I get an increasing sequence of simple functions. And phi n plus psi n converges to f plus g pointwise. And by this theorem that followed immediately from the monotone convergence theorem, I get that the integral of f plus g over E-- this is equal to the limit as n goes to infinity of integral of E of phi n plus psi n.

And now, we've proved linearity of the integral for simple functions. So this is equal to the limit as n goes to infinity of the integral of E of phi n plus the integral of E of psi n. And again, by the theorem that I just stated a minute ago, by the monotone convergence theorem, this converges through the integral of f, this converges through the integral of g. So the limit of the sum is the sum of the limits, and I get g.

OK. Using the same kind of argument, if you like-- except now not for two functions, but for one function-- you can show that the integral of a function over a union of two disjoint sets is the sum of the integrals. This is something that I pointed to but didn't prove at the very beginning of this lecture. But using that that is true for simple functions and this argument using the monotone convergence theorem-- which didn't require what I had proved earlier so this is not a circular argument-- you can prove that the integral of non-negative measurable function over a union of two disjoint sets is the sum of the integrals, one over the first set, one over the second set.

All right, so that's cool. This integral is linear over-- so the integral of the sum of two measurable functions is the sum of the integrals. What's even better is that the integral over an infinite sum is equal to the infinite sum of the integrals. So actually, let me state-- or let's just do this.

If fn is a sequence of non-negative measurable functions, then the integral of the sum of E is equal to the infinite sum of the integrals. Well, first off, this is a well-defined function because it's a sum of non-negative. So pointwise for each x, this is a sum of non-negative real numbers. So that's either going to be a finite number if this series converges or it's going to be infinite, all right?

Remember, we're allowing extended real value non-negative measurable functions in our framework for now. So this is meaningful. And it's a measurable function by stuff we proved in the section on measurable functions.

OK, so the proof is pretty short. By an induction argument and the previous theorem for the sum of two functions, we have the statement that for every fixed natural number of capital N, the integral of the sum, n equals 1 to capital N fn E is equal to the finite sum of the integrals. So if you can do something for two, usually you can do something for n by an induction argument.

So I'll leave the details of this to you or you can just believe it based on how many induction arguments you've done in your life. So we have this. And so since n equals 1, 1 fn is less than or equal to the sum from n equals 1 to 2 of fn is less than or equal to sum from n equals 1 to 3 fn. Because these are all non-negative functions, so adding non-negative functions to something only increases it.

And sum from n equals 1 to n converges to pointwise simply by defining this to be the limit as capital N goes to infinity of fn of x pointwise. All right, since I have these two things, then by monotone convergence theorem, I get that the integral of n equals 1 to infinity of fn of E. this is equal to the limit as capital N goes to infinity of the integral of E sum from n equals 1 to capital N, which, by what we have right here, is equal to the limit as capital N goes to infinity of-- now the finite sum comes out. And this is by definition this infinite sum.

So for non-negative measurable functions, the integral of the sum is equal to the sum of the integrals, even for an infinite sum. So again, this is simply false for if I replace everything by Riemann integration. Because, in fact, I can come up with a sequence of functions, fn, whose Riemann integral is always 0, but the sum is not Riemann integrable. Think of taking fn to be the function, which is 0 off of a rational number.

And then so first, enumerate the rationals Q1, Q2, Q3, Q4, and so on. And take fn to be the function that is 0 when x is not equal to Qn and 1 when x is equal to Qn. Then the infinite sum is just going to be the indicator function of the rational, say, in 0, 1. That's not Riemann integrable, but the sum of these integrals is just 0.

So this is not true for Riemann integrals again. So we're doing something much more powerful here. OK. Let's do some more properties of the integral.

Now, back to properties of the integral. So if I have a non-negative measurable function, then the integral of f equals 0 if and only if x equals 0 almost everywhere on E. So now what's-- this is a two-way street, so one direction.

If f is equal to 0 almost everywhere, then it's less than or equal to 0 almost everywhere. And therefore, the integral of f is less than or equal to the integral of 0, and the integral of 0 is 0. So this direction follows from the fact that f is less than or equal to 0 almost everywhere, which implies that the integral of f over E is less than or equal to the integral of 0, which you can check is 0. And this is a non-negative quantity, so.

So now, how about the other direction that the integral of f being 0 for a non-negative measurable function implies f is 0 almost everywhere. So let's let fn to be the set of all x's and E such that f of x is bigger than 1 over n. And let's let f be the set of all x's such that f of x is bigger than 0.

Now, if x is-- so this is x and E, I should say. Now, if I have an x where f of x is bigger than 0, then at least for some large n, f of x will be bigger than 1 over n So then the union from n equals 1 to infinity of the fn's equals f.

I mean, each of these is a subset of capital F, so their union is contained in capital F and I've just-- the argument I gave a minute ago shows you that capital F is contained in the union. So this union equals f. And just by how it's defined, 1 is bigger than or equal to a 1/2, which is bigger than the 1/3, f1 is contained in f2 is contained in f3 and so on.

If f of x is bigger than 1/2, it's certainly bigger than 1/3. Right, so now we'll use again, continuity of Lebesgue measure. So then for all n, 0, which is less than or equal to 1 over n times the measure of fn-- this is equal to the integral over E of 1 over n times the indicator function of fn.

Now, on this set-- but what am I saying? Yeah, let's write it this way. This is equal to 1 over n. And now, on fn, 1 over n is less than or equal to f of x.

So this is less than or equal to the integral of fn F. And capital F sub n is a subset of capital E, so this is less than or equal to the integral of E over E of f. But by assumption, this is 0, right?

And sandwiched in between 0 and 0 is 1 over n times the measure of f sub n. And therefore, for all n, measure of f sub n equals 0, which tells me that the measure of f-- which is equal to that union, which is equal to this increasing union-- is by the continuity of Lebesgue measure, equal to the limit as n goes to infinity of the measure of f sub n, which equals 0.

And therefore, the set of all x's where f of x is positive has Lebesgue measure 0. And f equals 0 almost everywhere. Now, using what we've done here and the monotone convergence theorem, we can slightly relax the assumptions in the monotone convergence theorem.

So we have the following theorem. If fn is sequenced in non-negative measurable functions such that now for almost every x in E, we have f1 of x is less than or equal to f2 of x is less than or equal to f3 of x and so on. And limit as n goes to infinity of fn of x equals a function f of x.

So remember, in the statement of the monotone convergence theorem, we assume these two things for every x. Now, we're just assuming them for almost every x in E. Then we get the same conclusion. Then the integral of E of f is equal to the limit as n goes to infinity of the integral of E over E of f sub n.

OK. So we call these two conditions star. Let capital F be the set while x is in E such that fn's are increasing to F, so star holds. Then the measure of the complement is, by assumption, equal to 0. And I should say here, the complement in E. So I should say E take away f.

So let me-- and that should have been in the-- so if I write complement, you should interpret that as the complement within E. So E take away f. Then f minus chi sub f-- so this is the indicator function over capital F. This equals 0 almost everywhere.

And fn minus chi sub f fn equals 0 almost everywhere for all n. These equal 0 when x is an f which the complement is as measure 0, which I didn't finish writing down. Now, by monotone convergence theorem applied now to these parts, if you like, and the previous theorem, we have that the integral of f of E, this is equal to the chi sub f.

And so this is equal to-- so since f minus f-- wait. Yeah, so OK. So since f equals f times chi sub f almost everywhere, the integrals equal. And so this is equal to the integral over capital F of little f.

And by the monotone convergence theorem applied here, this is equal to the limit as n goes to infinity of the fn's, because they are pointwise increasing on capital F to little f, and this is equal to-- OK, so I really didn't need the previous theorem. I could have used what I had earlier that if I have two functions which equal each other almost everywhere. So this previous theorem should not be referring to what I just proved a minute ago, but really to the theorem at the beginning of lecture that if I have two functions that equal each other almost everywhere, then their integrals equal each other.

Although, maybe I didn't state that. I just stated the less than or equal to. But if they're equal almost everywhere, they're less than or equal to each other almost everywhere. Anyways, back to this. This is equal to the limit fn.

So the whole point is that sets of measure 0 don't affect statements that involve integrals. That should be the take home, that if your conclusions are in terms of integrals, conditions holding almost everywhere suffice, typically. So for example, the simplest one we had earlier was that if f is less than or equal to g, then the integral of f is less than or equal to g.

We can relax that to the integral that if f is less than or equal to g almost everywhere, then the conclusion, which is stated in terms of integrals, still holds. The integral of f is less than or equal to the integral of g.

So now, we'll do the second big convergence of integrals-- or this one's actually an inequality between integrals, but it's still extremely useful. In fact, it's equivalent to the monotone convergence theorem, so it is neither stronger nor weaker. So we have Fatou's lemma, stated as a theorem, of course which states that if fn is a sequence in L plus of E, then the integral of E of the liminf is the integral that's infinity of fn of x.

This is a function. For each x, I take the liminf as n goes to infinity of fn of x. This is less than or equal to the liminf as n goes to infinity of the integrals of fn. So let me state it this way.

So the liminf of f sub n, let me just recall, what is the liminf? This is equal to the sup over n equals 1 inf over k bigger than or equal to n fk of x. So that is the definition of the limsup, if you like, if-- in fact, let me not just be specific to fn of x, just of a sequence of real numbers, the liminf of a sub n is equal to this thing on the right hand side.

OK. So this follows pretty easily from the monotone convergence theorem. I said a minute ago that it's, in fact, equivalent to the monotone convergence theorem. You can prove if you-- so we're going to use the monotone convergence theorem to prove it. You can also assume Fatou's lemma holds and then prove the monotone convergence theorem from it.

You can also prove it independently from the monotone convergence theorem. I mean, using essentially what's a similar argument to how you prove the monotone convergence theorem. OK, so first off, so we have liminf of fn of x, which is, again, by what I've written up here, sup n bigger than or equal to 1 inf k bigger than or equal to n fk of x.

This is now for a fixed n-- or what happens to what's in the bracket as n is increasing? Well, this inf is being taken over a smaller set. And the inf of a smaller set is bigger than or equal to the inf of the larger set. So this inf here, this thing in brackets, is increasing in n.

So this sup is, in fact, the limit as n goes to infinity of this increasing sequence of real numbers defined as the inf over k bigger than or equal to n of fk of x. And so what I just told you is not specific to fk of x. It's specific to ak, for sequence ak. OK, and basically, I'm going to write down what I said a minute ago.

Since fk bigger than or equal to 1 fk of x is less than or equal to inf a bigger than or equal to 2. We have fk of x is less than or equal to-- now it changes to 3 and so on. This implies by the monotone convergence theorem that the integral of the liminf of fn is equal to the limit as n goes to infinity of the integral over E of the inf k bigger than or equal to n fk. So I have this function here, which is defined in this way.

So for each n, I get a function here. All right. Now, for all j bigger than or equal to m, this function given by the inf over k bigger than or equal to-- let me add one more quantifier in here. So for all j bigger than or equal to n, for all x in E, I have that the inf over k bigger than or equal to n of fk of x-- this is certainly less than or equal to fj of x.

This is the inf overall fk of x for k bigger than or equal to n. And for any fixed j bigger than or equal to n, that's certainly less than or equal to fj of x. Because this is a lower bound for all of these guys for all j bigger than or equal to n.

And therefore, since this function here sits below this function, I have for all j bigger than or equal to n, the integral of E of inf of fk is less than or equal to the integral of fj. So I have this number here sits below this number here for all j. This is a fixed number depending on n, this is a fixed number depending on j. And this holds for all j bigger than or equal to n.

So this thing has to be a lower bound for the set of all numbers of this form for j bigger than or equal to n. And therefore, the integral of E of inf k bigger than or equal to n of fk is less than or equal to the inf overall j bigger than or equal to n of the integral of f sub j over E. Now, we're going to take this and stick it into this inequality here.

So that's what we had before, which was that the liminf of fn over E, which is equal to limit as n goes to infinity of the inf k bigger than or equal to n fk. This is, by what we've just shown, is less than or equal to the limit as n goes to infinity of the inf over j bigger than or equal to n of fj. But this is just, by definition, equal to the liminf of the integrals of the fn's, which is Fatou's lemma.

OK, so one more theorem about the Lebesgue integral, which is a very useful one. Throughout all this, we have had functions that are extended real value. So we're dealing with non-negative functions, which can equal infinity at points. And maybe that makes you nervous, but I'm going to tell you that as long as the integral is finite, you don't have to be nervous too often.

So if f is a non-negative measurable function over a measurable set E and the integral is finite, then the set of x is where f of x is infinite as the set of measure 0. So the measure of the set where it's infinite is 0. So what's the proof?

So it's kind of how we did, in spirit, the proof that if the integral is 0, then the function is 0 almost everywhere. So let f be the set of all x in E such that f of x equals infinity and fn be the set of all x's in E such that f of x is bigger than n.

Oh well, that's what I had in my mind, but what I wrote in my notes is a little bit different than the proof I had in my mind just now. So let's go with what's in my notes that's a little more cautious. OK. Then for all n, a natural number, n times chi f is less than or equal to f times chi f, where this is the indicator function of capital F. Because f on capital F is just infinite, so this always holds, right?

And therefore, n times the measure of f-- so of all n, n times a measure of f is less than or equal to the integral over E times f chi f, which is less than or equal to the integral of E of f, which is finite. That's a fixed number.

Then for all n, the measure of f is less than or equal to 1 over n times the integral of f over E. Again, this is a fixed finite number which goes to 0 as n goes to infinity. This is just a fixed number as well. Thus, measure of f equals 0.

OK, so that seems like a good place to stop. Next time-- so we've defined the Lebesgue integral of a non-negative measurable function. We will then define the class of Lebesgue integrable functions and extend the definition of integral to those functions in a fairly straightforward way.

Prove some simple properties of the Lebesgue integral. And also, the last big convergence theorem, which is the dominated convergence theorem. And we may or may not finish by the end of next lecture the proof that Lp spaces, which are based on the Lebesgue integral-- so we built the Lebesgue integral to have a space of functions for which kind of-- OK, so let me stop. That alarm kind of threw me off.

So it's not too difficult to show proof. Or you can just accept for now-- and we'll actually see why this is the case soon-- that the space of continuous functions with norm being the integral. So the norm of f being, let's say, the integral of the absolute value of f is a norm space, but it's not a Banach space. Or you could change the integral of the absolute value of f to what's called a big Lp-norm, the integral of f raised to the p all raised to the 1 over p. So the analog of the little Lp norms, which we encountered a few weeks ago.

None of those are Banach spaces when restricted to continuous functions, or even Riemann integrable functions. So our goal-- at least it was a while back when we started this section, this big section on Lebesgue integration-- was to build, or at least come upon a space where the resulting integrable functions form a Banach space. And so we may or may not, by the end of next lecture, introduce those.

But that's where we're headed. That's where we're almost at. And these spaces arose because one wants to apply functional analysis facts, tools to concrete questions, such as questions about convergence of Fourier series, which arose immediately after Fourier said that any periodic function can be expanded as a Fourier series. So a lot of people went to a lot of trouble to fill in precisely what it means expanded as, expanded as pointwise.

Does this Fourier series converge to the function that was kind of hard to do on average? Do you mean average as in measured with respect to some norm that's integrated that involves integration? So which is why we're coming here. But we'll see that next time. Or we'll see the applications of this integration theory, along with the functional analysis later in the course when we circle around to Fourier series. All right, so we'll stop there.