[SQUEAKING]

[RUSTLING]

[CLICKING]

CASEY RODRIGUEZ: OK, so let's continue our discussion about spectral theory for self adjoint compact operators. So let me just briefly recall the spectrum of a bounded operator, which was supposed to be a generalization of the eigenvalues of a matrix.

So we defined the resolvent set of A to be those complex numbers such that A minus lambda times the identity, which I just write is A minus lambda, is an invertible, bounded linear operator, meaning it is bijective and, which by the open mapping theorem, tells you that the inverse is also continuous.

And the spectrum of A is simply those lambda so that A minus lambda is not invertible, so the complement of resolvent set of A. So from linear algebra, you have the following characterization of what the spectrum is, that if H is CN, and A is just therefore matrix on CN, or RN if you like, then the spectrum is just simply the set of eigenvalues of A.

And if we restrict our attention to Hermitian matrices, which are referred to as-- which are also referred to as self adjoint also in linear algebra, or symmetric if we're just looking at real vector spaces Rn then the eigenvalues are real.

And you can find an orthonormal basis of the space CN or RN, depending on what you're looking at for this symmetric matrix so that in that basis, the matrix is completely diagonalized. and. What are the diagonal elements there? The eigenvalues of the matrix.

And what we're going to end up proving is that that picture, this picture where for a self adjoint matrix on CN, that the spectrum is given by the eigenvalues. And you can diagonalize, essentially diagonalize this operator, meaning you can find an orthonormal basis consisting entirely of eigenvectors of the operator is also true for self-adjoint compact operators.

This shouldn't come too much of a surprise, as too much of a surprise, since compact operators are limits of finite rank, i.e. matrices in the space of bounded linear operators. So that's where we're headed.

Now, so just to follow up on this and for a little bit of review, so in the finite dimensional case, the spectrum can be-- or the spectrum is always just the eigenvalues of the matrix A-- not so in the infinite dimensional setting, for example, if we're looking at little l2. And then A times a is equal, to let's say, a1 over 1, a2 over 2, and so on, for a, a sequence in l2.

Then what you can prove is that 0 is in the spectrum of A. One way to see that is that each of the basis vectors of H given where you just have 1 in the n-th slot and 0 otherwise, each of those is an eigenvector of this operator A with eigenvalue 1 over n, where the n tells you where the 1 is and 0 otherwise.

So 1 over n is an eigenvalue of this operator for each n. And 1 over n convergence to 0. And since the spectrum of a bounded linear operator is a compact set, in particular closed, 0, which is the limit of that sequence, has to also be in the spectrum.

And, in fact, what we're going to show is that in the nondegenerate case, what we see here is what in general happens for compact self-adjoint operators, that if it's not a finite rank operator, then it has countably many-- or countably infinite many-- that's not a very good string of words-- countably infinite eigenvalues which converge to 0. And 0 may be an eigenvalue, may not be.

And what's more, that's going to be the only that completely characterizes the spectrum. So let me just write here for this example that, in fact, the spectrum of A is equal to the point 0 union 1 over n in a natural number.

And these are-- again for this operator here, these are the eigenvalues. And this is just-- well, 0 is 0. But 0 is not an eigenvalue, OK. Now, this is not the general picture, meaning that for a compact self-adjoint operator, you'll have infinitely many eigenvalues. And then 0 will not be an eigenvalue. You could have 0 in eigenvalue as well.

But in general what the picture is is that anything in the spectrum that's not 0 has to be an eigenvalue. And that's essentially what will prove our first result of this lecture is the following. So this is the Fredholm alternative. So let A be a self-adjoint compact operator and lambda be a nonzero real number.

Then the range of A minus lambda is closed. So this is the conclusion. And thus the range of A minus lambda is equal to the orthogonal complement of the orthogonal complement of itself, which-- the orthogonal complement of the range of A minus lambda is equal to the null space of the adjoint. And since A is self-adjoint and lambda is a real number, the adjoint is just A minus lambda.

So this right here is the main conclusion from which you get this. And therefore just to spell out the conclusion of this equality, therefore either one of two alternatives happens. So that's the name, alternative-- either A minus lambda is bijective, or the null space of A minus lambda, which is just the eigenspace of A minus lambda, or I should say the eigenspace corresponding to lambda, is non-trivial and finite dimensional.

Moreover, this equality here tells you when you can solve the equation A minus lambda u equals f. You can solve a minus lambda u equals f if and only if f is in the range of A minus lambda, which is if and only if f is orthogonal to the null space of A minus lambda. So let me make a little remark here.

So first off, the fact that this space has to be finite dimensional we proved last time. We proved at the end of lecture, the last lecture, that for a compact self-adjoint operator, the null space, the eigenspace corresponding to a given nonzero eigenvalue, is finite dimensional. And then we also prove the eigenspaces corresponding to two different eigenvalues are orthogonal. And we also prove that the eigenvalues that are nonzero, or any eigenvalue, has to be real.

So now let me make a couple of remarks. The first is just a rephrasing of what's in the theorem. Therefore f is in the range of A minus lambda, meaning you can solve the equation A minus lambda u equals f if and only if f is in the null space of-- or the orthogonal complement of A minus-- or the null space of A minus lambda.

So what this says is that you can solve for f, given that f satisfies finitely many linear conditions because the null space of A minus lambda is finite dimensional. So f being orthogonal to that means pick a finite base, a finite orthonormal basis of A minus null space of A minus lambda. Then f in our product with those finitely many vectors has to be 0.

So you have finitely many conditions on f to be able to solve for f. And not only that, this solution that you compute is also unique up to finitely many conditions because, again, the null space of A minus lambda is-- or unique up to a finite dimensional subspace, again, because the null space of A minus lambda is finite dimensional.

And the second is that for a self-adjoint operator, we have that the spectrum is a subset of the real numbers. For a self-adjoint operator, this is something we proved last lecture. It doesn't have to be compact, just a self-adjoint operator.

So since the spectrum is a subset of the reals, this proves that for a compact self-adjoint operator, the spectrum of A is equal to the set of eigenvalues of A, or I should say nonzero eigenvalues of A. Let's write it this way.

If I look at what's in the spectrum other than 0 possibly, then the nonzero numbers that are in the spectrum have to be eigenvalues. So for a compact self-adjoint operator, and just in the case of matrices, the spectrum, the nonzero spectrum has to be-- are nothing but eigenvalues.

And last time, remember, we proved that the eigenvalues are countably infinite, or countable. They're either finite or countably infinite. And if they're countable infinite, they converge to 0. So, again, by last lecture, we conclude that the spectrum of A take away 0 equals either finitely many eigenvalues or countably infinite eigenvalues that are converging to 0.

So from the Fredholm alternative we get a lot of information about when we can solve equations. But it also tells us-- I mean from that ability to say when we can solve the equations, we can also characterize the nonzero spectrum of a self-adjoint compact operator.

So we need to prove that-- so remember all of this just followed from stuff we had proven and the main conclusion of the theorem, which is that the range of A minus lambda is closed. So we need to prove that the range of A minus lambda is closed when lambda is a nonzero real number.

So suppose that you have a sequence in the range, which I'll write as A minus lambda times un converging to some element f in H. So what we'd like to be able to show is that f is in the range. So we want to show f is in the range of H-- or not range of H, range of A-- minus lambda, sorry. Hopefully I didn't make that mistake elsewhere-- no, just-- OK.

Now, we're only assuming that a minus lambda when it hits u sub n converges to f. We're not a priori assuming the use of n's converge. In fact, we can't. But in the end we would like to come up with maybe a subsequence or a part of the u sub n's up to a subsequence which does converge, and then we conclude that f is in the range.

So first I want to get rid of the useless part of the u sub n's. So let W be-- well, I don't need to give it a name. Really, where is my eraser? So let vn be the projection onto orthogonal complement of the null space of A minus lambda of u sub n.

So now this is just part of-- so every u sub n is written-- you can write as something in the null space of A minus lambda. So since null space of A minus lambda is a closed subspace of H, it has an orthogonal complement so that it and its orthogonal complement gives the direct product-- or when you take their direct product, gives you H.

So why am I saying that? Because then if I take A minus lambda u sub n, this is equal to A minus lambda applied to pi, so the projection onto the null space of u sub n plus the projection onto the orthogonal complement of the null space, which I defined as v sub n.

Now, this element here is in the null space of A minus lambda. So when A minus lambda hits it, I get 0. So I get A minus lambda applied to v sub n, all right. Then a minus lambda v sub n equals A minus lambda u sub n, which converges to f.

So, basically, I've taken away some noise, all right, the part that when it hits A minus lambda, I get 0. So now I just have these v sub n's which lie in the orthogonal complement of the null space of A minus lambda.

So my claim is, first, is that the sequence v sub n is bounded. Basically, once I can show this, then I'm done because if I can show v sub n is bounded, then since A is a compact operator, when A hits v sub n up to a subsequence, that converges.

Now, this whole expression converges and therefore lambda times v sub n converges. Lambda is nonzero, so then v sub n converges up to a subsequence to something. And therefore A minus lambda v sub n then converges to A minus lambda v for some v, which shows that f is in the range of A minus lambda. So this is really the whole ball game.

And we'll use here crucially as well that we threw away useless parts of u sub n, useless at least to this argument. So I claim this is bounded, so suppose not. Then there exists subsequence v sub n sub j such that v sub n sub j goes to infinity as j goes to infinity.

All right, now if I look at A minus lambda applied to v sub n sub j over norm of v sub n sub j, this converges-- so first off, since is a linear operator, this is equal to 1 over norm v n sub j times A minus lambda applied to v sub n sub j. And so this scalar, 1 over norm v n sub j converges to 0. This converges to f. So I get the 0 vector in H.

So this thing converges to 0 in the Hilbert space H. Now, why is that bad? Because essentially what this is going to say is that there exists some element, or that this sequence converges at least up to a subsequence to an element v with norm 1 because they all have norm 1 so that A minus lambda v equals 0. But all of these are in the null space of A minus lambda and we get a contradiction.

So we have that A minus lambda-- so we have that part. Since is A is a compact operator, there exists a subsequence, so a further subsequence-- I'm just going to call it n sub k instead of n sub j sub k-- v sub n sub k of v sub n sub j such that the sequence a sub v sub n sub k converges.

But then I get that v sub n sub k-- or I should say v sub n sub k over norm of v sub n sub k. So these all have norm 1. And A applied to something that has unit length-- or the image by A of the closed unit ball is a precompact-- or the closure of it is compact. And therefore every sequence has a convergent subsequence.

Then v sub n sub k over norm of v sub n sub K, , this is equal to 1 over lambda times A applied to v sub n sub k over norm of v sub n sub k minus A minus lambda applied to v sub n sub k. Now this sequence of elements converges to 0. That's, in fact, what we just proved. And this converges by how we've taken this subsequence because A is a compact operator.

So I have this sequence of vectors is equal to-- and here we can divide by lambda because lambda is nonzero. It's equal to a linear combination of two sequences which converge. And therefore we get that v sub n sub k over norm of v sub n sub k, k converges to an element v.

And now the null space or the orthogonal complement of the null space of A minus lambda, each of these is in-- remember the orthogonal complement of the null space. And since it's converging to an element and this is closed, this element has to be in the same set. So, again, this follows from-- the fact that v is in here is because this set is closed. The orthogonal complement of any subset of a Hilbert space is closed.

Then by continuity of the norm, basically, the norm of v has to be equal to limit as k goes to infinity of the norm of the elements converging to it, which all equal 1. And if I compute A minus lambda applied to v, this is equal to-- since the v sub n sub k's over norm v sub n sub k's are converging to v A minus lambda applied to v sub n sub k over norm v sub n sub k.

And, remember, this is a subsequence of v sub n sub j's. And when A minus lambda hits that, they're converging to 0, so-- all of that predicated upon the assumption that the norm of v sub n sub k's converges to infinity or that the sequence is unbounded.

So we have this element in the null space that has-- or the orthogonal complement of the null space that has norm 1 but also gives you 0. And therefore we get that v is in the null space of A minus lambda from this computation and it's orthogonal complement.

But the only possible vector that's in a space and it's orthogonal complement is the 0 vector, or the only vector in a subspace and its orthogonal complement is the zero vector. And therefore v equals 0, which this is a contradiction to the fact that the norm of v equals 1.

So we started off with the sequence v sub n's. Assuming that they are unbounded, v sub n over norm v sub n is a sequence of essentially-- yeah. So maybe you got lost in the subsequences. But let's just assume I'm talking about the entire sequence.

Then v sub n over norm v sub n, when a. Minus lambda hits it, converges to 0. Since A is a compact operator, we can show essentially that A applied to v sub n over norm v sub n, because those things have unit length, converges to something.

And since lambda is nonzero, we can then conclude that those vectors converge, in fact, to something, not just their images by A or their images by A minus lambda, again, because lambda is nonzero.

And since they all have unit length, their limit must have unit length. And since when a minus lambda hits these guys they go to 0, the limit must also, when A minus lambda hit it, equals 0. And that gives us our contradiction because this limit v had to be in the orthogonal complement of the null space, but then also in the null space, and have unit length. Those three things can all happen at once.

So thus the sequence v sub n is bounded. So, remember, what were the v sub n's to start with? They were so that A minus lambda v sub n's converged to this element f. And we wanted to show that f is in the range.

So we coming back over here. We had these v sub n's so that A minus lambda v sub n converges to f. And we want to show f is in the range to conclude that the range of A minus lambda is closed. But now since it's bounded and we've done this argument already, this is essentially the whole ball game.

Since the subsequence v sub n is bounded, and A is a compact operator , we conclude that there exists a subsequence v sub n sub j such that-- so this has nothing to do with the previous argument now, but I just don't feel like using different letters-- such that a applied to v sub n sub j converges.

So, remember, is A compact operator, which we stated it in terms of the closure of the image of the closed unit ball being compact. Equivalently by scaling the unit ball, it means that A takes any bounded sequence to a sequence that has a convergent subsequence.

So we showed v sub n is bounded. And therefore since A is a compact operator, we can find a subsequence so that when A hits it, we have a convergence sequence. And by that same trick we used a minute ago, we conclude that v sub n sub j, which is-- and we can do this because lambda is nonzero. We can divide by it.

So A v sub n sub j minus A minus lambda V sub n sub j-- now, again, this here is converging to something. This here is converging to f. So this linear combination of convergent sequences is convergent-- converges to and element v.

And therefore I get that f which is the limit as n goes to infinity of the v sub n's. But convergence still holds if I look at a subsequence, A minus lambda v sub n sub j equals-- and since A-- so A is a bounded linear operator. Lambda times the identity is a bounded linear operator. This is equal to A minus lambda v. And therefore f is in the range of A minus lambda.

OK, so Fredholm alternative tells you that the range of A minus lambda for a compact self-adjoint operator is closed. So where did we really used the fact that it was self adjoint? Nowhere in this argument. So this fact that the range of A minus lambda is close is still true if A is just a compact operator and lambda is just some nonzero complex number.

But where we use that it's self adjoint is I guess in the rest of the conclusion, that the range of A minus lambda is therefore equal to the null space of A minus lambda. And therefore either A minus lambda is bijective, or the null space of A minus lambda is non-trivial and finite dimensional by what we did in the previous lecture.

OK, now, again, this is a very powerful theorem. And, again, what this says is that if I look at the nonzero spectrum of a compact self-adjoint operator, then that consists entirely of eigenvalues of A.

Now earlier we proved that plus or minus the norm of A has to be in the spectrum of a self-adjoint operator. And therefore what we can conclude is that if we have a non-trivial self-adjoint compact operator, then it has at least one eigenvalue.

And we can characterize that eigenvalue. So this has a following theorem. Let A be a non-trivial compact self-adjoint operator, A equals A star. Then has a non-trivial eigenvalue lambda 1.

And we can characterize lambda 1-- or at least the absolute value of lambda 1-- as the supremum over norm u equals 1 Au, u equals-- and this supremum is actually achieved where u1 is a normalized eigenvector corresponding to lambda 1.

So why do we have-- or I should say, let me make sure I have everything here. OK, so why is this? So first off, we've shown that plus or minus the norm of A is, in fact, in the spectrum of A. for any self-adjoint bounded linear operator, not necessarily compact, that plus or minus-- not necessarily both of them, but plus or minus, one of these, at least one of plus or minus, since a is self adjoint, meaning A star equals A.

Then lambda 1-- then I'm going to say plus or minus, meaning not both, actually at least one of these, is an eigenvalue of A by the Fredholm alternative. The Fredholm alternative and the fact that-- so lambda one is going to be either plus norm of A or minus norm of A, depending on whether which one is in the spectrum. Let's say plus if it's in the spectrum, minus if plus is not.

And the fact that we can identify it as this quantity here is because we have that for self-adjoint operators, norm of A, which is the absolute value of one of those plus or minuses that are in the spectrum is equal to-- so this is where-- so this is equal to sup u equals 1, Au equals u.

So that's the end of the proof. And now what we're going to do is we are going to keep going. So the end result will be that we can determine all of the eigenvalues via a certain maximum principle and build up a sequence of eigenvectors, normalized eigenvectors, which are pairwise orthogonal simply because they come from how they're built up. But we'll see.

And what? And then we'll show that essentially that set of eigenvectors that we get, along with a set of-- or an orthonormal basis chosen for the null space of A, form an orthonormal basis for a separable Hilbert space H.

So that's where we're going. But we can basically take this theorem and keep applying it. So we have the following maximum principle. If you like, this is the first step in a maximum principle. This says if you want to find the largest eigenvalue, lambda 1-- I should say-- I can even say largest eigenvalue, largest in the sense of absolute value.

Why? Because remember the spectrum is contained in the interval minus norm A plus norm of A. So anything in the spectrum has to have absolute value less than or equal to the norm of A. And if it's eigenvalue, or if it's a-- anything other than 0 has to be an eigenvalue so we get this.

So what's this maximum principle? And why was I saying all that? Oh, so this gives you a way to-- if you'd like try and find or at least approximate the first eigenvalue of a bounded linear operator, this is a maximization problem with a constraint. So you could use the method of Lagrange multipliers.

That doesn't maybe make sense to you to be able to do on an infinite dimensional Hilbert space. But let's say you choose a big basis of your Hilbert space. And you just restrict to looking at that big but finite dimensional, or finite basis, or span of that finite basis, and try to solve the approximate problem, then you should get close to the eigenvalue and get an approximate eigenvector because as I've said, the eigenvector will be achieved-- or I should say that the eigenvector achieves this maximum here , or the supremum.

So the maximum principle is the following. So let A-- again, we're only looking at compact self-adjoint operators, compact operator-- then the nonzero eigenvalues of A can be ordered.

Now, this part we already know. They can be ordered lambda 1 less than or equal to lambda 2, less than or equal to lambda 3, counted with multiplicity meaning if lambda 1 has a two-dimensional eigenspace then lambda 2-- or the absolute value of-- or lambda 2 will be lambda 1. So we'll repeat it according to multiplicity.

So we know that there's finitely many distinct eigenvalues. That was the part that I was saying. We know we can order them, but maybe it's not clear that you can order them with multiplicity with corresponding orthonormal eigenbasis functions uk. So u1 is a normalized eigenfunction for lambda 1. u2 will be a normalized function for lambda 2, which is orthogonal to u1. So these are pairwise orthonormal.

And how do we obtain the eigenvalues in this order and these eigenfunctions via the following process so that lambda j is equal to the supremum over all unit vectors that are orthogonal to the first j minus 1? And this is equal to-- this will be achieved on the u sub j.

So we have the first one, if you like. We built up lambda 1, u1. and. Now what this maximum principle theorem is saying is that we can repeat this, is that if we now look at this quantity here, the supremum over all norm 1 vectors that are orthogonal to u1, then we're going to pick up the next largest eigenvalue counted with multiplicity, meaning it'll be the absolute value of lambda 2, which will be less than lambda 1 if lambda 1 only has a one dimensional eigenspace.

Or it'll be lambda-- or the absolute value of lambda 2 will be the absolute value of lambda 1 again if lambda 1 has a, let's say, two-dimensional eigenspace. And if it had a three-dimensional eigenspace, we would get the absolute value of lambda 1 for-- it would equal this number and this number as well. So, again, these nonzero eigenvalues can be ordered in this way. Oh, and I left off one. And the lambda j is going to 0.

OK, so we have the first one, the absolute value of lambda 1, and the first eigenvector u1. and. Now we're just going to basically repeat or apply the previous theorem to a modification of A. Now I should say this is not entirely true because there may only be finitely many.

So really this should be a remark, if the decreasing sequence does not terminate, then the absolute value of the lambda j's goes to 0. Now, so what's the new bit of information here? We do know that we can-- for each, let's say, capital N, the number of eigenvalues outside of-- or with absolute value bigger than 1 over N has to be finite.

So the fact that we can order them is not really that much new information. And the fact that they to go to 0 if this sequence is infinite, that's also not new information. I mean, we did prove that in a previous lecture that if you have-- so there we proved it for a sequence of distinct eigenvalues.

But because we now know that each of these has a finite dimensional eigenspace, if you just look back at the same proof or that proof, you can make a small adjustment to be able to say that if you count the eigenvalues with multiplicity, then also the sequence has to go to 0, not just necessarily the sequence consisting of the distinct eigenvalues. So let me make that small remark.

What the new piece is that we can compute the eigenvalues in this way and choose the eigenvectors in this way. That's a. New piece and that's what, in the end, we'll use to be able to show that H can be-- or that A can basically be diagonalized, or that you can find an orthonormal basis of a Hilbert space, separable Hilbert space consisting entirely of eigenvectors of A.

So the construction proceeds inductively. So at one point I said, if you can d it for one then you can do it for the next. And then I won't be so formal every time I do an inductive construction. But in this case it pays to be careful. So we'll construct the sequence of lambda j's and uk's in this way via an inductive argument.

So coming up with k equals 1, that was the previous theorem. So previous-- or I should say j equals 1, that's the previous theorem. We found the largest eigenvalue, or the eigenvalue with-- an eigenvalue with the largest absolute value via this theorem. And then we obtained an eigenvector this way, just basically as a consequence of the Fredholm alternative. And then that we know that plus or minus the norm of A has to be in the spectrum.

| the fact that we can find the first eigenvalue of lambda 1, that follows from the previous theorem. So now we want to do the inductive step, i.e. we want to suppose we have found lambda 1 up to lambda-- let's say, what did I use here?

Lambda n, so j equals n and with orthonormal eigenvectors u1 up to un, satisfying this maximum principal. So j starting from-- starting at 1 up to n, we've constructed or found a lambda 2, lambda 3, up to lambda n, and the u went up to un, satisfying that maximum property.

So then there's two cases. So in case 1 is the fact that A minus-- A is equal to the sum from k equals 1 to n of lambda k, u inner product uk, uk. And therefore what this shows is that we found all the eigenvalues and the process terminates.

So this is kind of the degenerate case that A is a finite rank operator. So I could have started this whole theorem off with let's assume A is not a finite rank operator, and therefore we wouldn't have to deal with this case. But I just stated it for an arbitrary A.

So there is the possibility that that sequence stopped. We found all of the eigenvalues with multiplicity in this process. And then the theorem-- or construction is done in that case. And the case 2 is that it is not a finite rank operator, so it's not equal to k equals 1 to n.

Now we have to show how can we find lambda sub n plus 1 and the eigenvector u sub n plus 1. Let a Sub n be the operator A minus k equals 1 to n lambda k inner product u. So should say Au minus u sub k, all right.

Now. Note since we are in case 2, this operator is nonzero. So we're basically going to apply the previous theorem now to this operator. But let me first make a few remarks. Then A is A sub n is a-- this is something you can check. It's a self-adjoint compact operator. So why is it self adjoint? Basically because A is self adjoint and these lambda k's are real numbers because eigenvalues have to be real, because A is self adjoint and these are orthonormal.

So that is why they're self adjoint. Why is it a compact operator? Well, it's the sum of a compact operator A and a finite rank operator here. So it's also a compact operator. And it's a nontrivial one because it's not identically equal to 0. So I shouldn't say not equal to 0, but which is not identically the zero operator.

So here's a couple of facts that if u is in the span of u1 up to un, then I get that A sub n applied to u is the zero vector. Why is that? So it suffices to check that this gives me-- this formula gives me 0 when u is one of the uk's.

Now if u is one of the uk's, then this will be 1 only when-- or let's say it's one of the uj's. I need a different letter. So if u is one of these uj's, then this will give me 1 only when j equals k and I pick up lambda j times uj. And then I have A hitting u sub j. And that spits out lambda j times u sub j because u sub j is an eigenvector of A. So then I get the same thing here and there and subtracting them gives me 0.

Now, if u is orthogonal to the span of the u1 up to un, then I get A n of u equals Au. So you just see this. If u is orthogonal to these, then this whole term is 0. Can't I just pick up An u equals Au.

For all u in H, in the span of u1 up to un, if I compute A and u inner product v, this is equal to u and v since a n is self adjoint. And since v is in the span of the u1's up to un's, by the first property that's going to be 0.

So this equals 0 and therefore, what have I shown? I've shown that A and u inner product v is 0 no matter what u is in H. And therefore, that means that v has to be in the orthogonal complement of the range. So I said that backwards.

Anyways, here u is fixed. v is a thing that's changing. So what does this say? u as fixed. A and u inner product v when v is in the span has to be 0. This holds for all v in the span. And therefore A n u has to be in the orthogonal complement of the span of these guys. So this proves that the range of A n is a subset of your orthogonal complement of u1 up to un.

And so from this previous property we get one last property, that if a n u equals lambda u is nonzero, meaning we have a nonzero eigenvalue of A sub n, then that implies that u-- so u is equal to 1 over lambda A sub n appliec to u.

In other words, u is equal to A sub n applied to u over lambda. So that implies u is in the range of A sub n, which again is contained in the orthogonal complement of u1 up to u sub n. And since it's in the orthogonal complement, that means A sub nu equals Au. So any nonzero eigenvalue of A sub n has to be a nonzero eigenvalue of A.

Now we just apply the previous theorem to A sub n to the next eigenvalue lambda sub n plus 1 and eigenvector u sub n plus 1. By the previous theorem, a sub n as a nonzero eigenvalue, which I will call lambda n plus 1, with unit eigenvector u sub n plus 1 so that lambda n plus 1 is equal to the sup over all norm-- ah, OK, sup over all norm or unit length vectors of absolute value of A sub n applied to u, inner product u.

Now, thos sup is the same sup over all 1 so that A sub n u is nonzero. So that in particular includes-- OK. So if u is in the span u1 up to un, then A and u equals 0.

So the supremum over all unit length is the same as u-- the supremum over everything in the orthogonal complement of these guys because these guys, when you stick them into A, and gives me a 0. So that's the same supremum.

But now also when u is in the orthogonal complement of these u1's up to un, this is equal to sup of u on the orthogonal complement of the u1 up to un's. A n u, remember, equals Au. So that gives us the fact that I can find this from this supremum.

And so why does this eigenvector has to be in the orthogonal complement of u1 up to un? it's. Because I can choose it that way because when A sub n hits anything in here, I get 0. So I should say this is also equal to A n apply to un plus 1. But that's the same as A applied to un plus 1-- un plus 1.

And this is less than or equal to the sup over norm u equals 1 u in span u1 up to un minus 1, orthogonal complement Au applied to u, which equals the absolute value of the n-th eigenvalue counted with multiplicity. So we found the next eigenvalue in this sequence of eigenvalues countable multiplicity.

So now, we can conclude the following spectral theorem for compact self-adjoint operators. Let A be a self-adjoint compact operator on a separable Hilbert space H.

Let lambda 1 bigger than or equal to lambda 2 be corresponding eigenvalues, be the-- eigenvalues-- or I should say nonzero eigenvalues-- of A counted again with multiplicity, so counted with multiplicity, as we've constructed in this theorem which I called the maximum principle, with corresponding orthonormal eigenvectors uk.

So we have these eigenvectors coming from this process. And the conclusion is this subset of eigenvectors, or the orthonormal eigenvectors is an orthonormal basis for the closure-- or for the range of A. In fact, we can upgrade that the uk is, in fact, an orthonormal basis for the range of A closure.

And there exists an orthonormal basis, call it f sub j, of the null space of A if it's nonzero, such that uk k fj. So, first off, the union of these two sets of orthonormal vectors is then going to be again a subset of orthonormal vectors because all the fj's would correspond to the eigenvalue 0. And these uk's would correspond to eigenvalues that are nonzero.

And by what we proved last time, any eigenvector for two distinct eigenvalues would have to be orthonormal. So this subset is orthonormal from this subset. But, moreover, is orthonormal like I said, but also an orthonormal basis for H.

So, in other words, I can find an orthonormal basis for H consisting entirely of eigenvectors of this self-adjoint compact operator, OK. So I will have one piece of this basis coming from the null space and the other piece corresponding to nonzero eigenvalues.

So, really, 2 follows from 1. Not sure why I decided to state them separately, but here we are. So proof of 2 will show that uk is in orthonormal basis for the range of A.

So first off note as we did in the previous proof that the process of obtaining the lambda 1 for the eigenvalues and eigenvectors, or orthonormal eigenvectors terminates if and only if A was finite rank.

In other words, their exists an n so that Au is this finite rink operator u in a product uk, uk. In this case, if it's a finite rank operator, then the range is contained in uk, uk, which is what we wanted to prove for this case, that A is this finite rank operator.

So suppose otherwise, in other words the process does not terminate, so that we have countably infinite many nonzero eigenvalues counted with multiplicity and corresponding orthonormal eigenvectors u sub k. So this is a more interesting-- so the eigenvalues are countably infinite.

OK, so now we want to show-- so as in the remark that I made afterwards, you know that these lambda k's have to go to 0. Now, we want to prove that the uk's are an orthonormal basis for the range of A. What does that mean?

By definition, that means that they're in maximal orthonormal subset of the range of. A So we have to show that if something's in the range and it's orthogonal to every one of these eigenvectors, then that thing has to be 0. So the claim if f is in the range of A, and for all k f inner product uk equals 0, then f is a 0 vector.

So this is the claim we want to prove. So suppose we have something in the range. That means we can write it as A times u and f inner product uk equals 0 for all k. Then for all k, if I look at lambda ku inner product uk, this is equal to-- lambda k is a real number so I can bring it all the way in, get lambda k, uk. And this is equal to u A apply to uk.

Now, a is self adjoint so I can move this a over here to u. And this is equal to f inner product uk equals 0 for all k. So by this maximum principle which we proved just a minute ago we conclude that the norm of f, which is equal to the norm of Au, which is equal to the norm of A minus sum from k equals 1 to n of lambda k u, uk, uk applied to u because every one of these numbers is 0, so I haven't subtracted off anything.

So I can write this in terms of this A sub n applied to u, where A sub n was this thing-- or A sub n is this thing, which by the proof of the maximization-- or the maximum principle is less than or equal to lambda plus 1 n plus 1 of u because, again, remember this thing here is the supremum over all u's of unit length of A sub n applied to u.

So lambda n plus 1 is less than-- or this quantity here divided by the norm of u so that you as a unit length is always less than or equal to this quantity here. But now lambda-- so this-- I had a fixed thing here, norm of f. And I've shown it's less than or equal to lambda n plus 1 times the norm of u. This is a fixed thing. And the lambda n's are converging to 0.

And thus I started off with something non-negative, less than or equal to something converging to 0. And therefore that thing had to be 0. Therefore f is 0. So this proves, 1, that these eigenvectors are a maximal orthonormal subset of the range of A.

And for 2 we simply note that by 1, we have that the range of A closure, this is since the eigenvectors are in orthonormal basis for the range of A, the closure is contained in-- closure of the spans of the uk's, here this is a finite span, which, remember, this is by an assignment exercise, this is equal to k ck, uk such that-- and therefore, this implies that is an orthonormal basis for the range of A closure.

And this is equal to the range of A orthogonal complement, orthogonal complement, which is equal to the null space of-- or the orthogonal complement of the null space of A star. A star equals A, so we get that.

So the uk's-- the eigenvectors form an orthonormal basis for the orthogonal complement of the null space of A. So once we've chosen orthonormal basis for null space of A, that's it. Since H is separable and the null space of A is a closed subspace of H, null space of A inseparable.

And we've proven that every separable Hilbert space, or even just a-- so this is a close one so it's a Hilbert space anyways. But every separable Hilbert space has an orthonormal basis of-- and therefore, again, so since these two-- call this fj-- this is an orthonormal basis for the null space of A direct product null space of A orthogonal complement. That equals H.

And just in the nick of time I'm finished. So next time where do we go from here? We'll see some of this applied in a concrete setting of differentiable equations and also discuss the functional calculus.