[SQUEAKING]

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[CLICKING]

So let's continue with the big-name theorems, or the theorems with big names, or the big theorems with names, which the last one was a uniform boundedness theorem, which followed from Baire's category theorem, which let me again recall for you. If M is a complete metric space, C k is a collection of closed sets and M is equal to the union of these closed sets, then at least one of these closed sets has an interior point. Or another way of saying it, at least one of these C k's contains an open ball.

So now, what we're going to prove as a consequence of the Baire category theorem is so-called "open mapping theorem," which says that surjective linear, bounded linear operators send open sets to open sets. It's kind of the backwards version of continuity. So if B1, B2 are Banach spaces and T is a bounded linear operator from B1 to B2, which is surjective, meaning onto, then T is what's called an "open map."

Or you would just say, "T is open," which means for all open u, for every open subset of B1, T of u is open in B2. So T takes open sets to open sets. So statement of the theorem-- if you have a bounded linear operator between two Banach spaces which is surjective, then T is an open map.

So we're first going to specialize to one type of open set. And we're going to prove a specialized version of what we want to prove. And then we're going to show using the linearity of T, along with scaling and translation, that this, then, implies that open sets get mapped to open sets.

So first, what we're going to prove is that-- let me recall this is the set of all B in B1 such that-- so this is the open ball in B1-- then the image of this open ball contains an open ball. So now, the image is in B2, an open ball in B2 centered at 0. So T of the open ball could be some of weird set, and we know that linear maps take 0 to 0.

What we're saying is that there exists an open ball contained in the image. Now, like I said, once we prove this, we're going to use linearity of T to be able to shift these balls around, shift and scale these balls around so that we prove the result for every open u. But this is the heart of it.

So since T is surjective, everything in B2 gets mapped onto. So B2 is equal to the union over N natural number of T of B closure. So everything in B2 gets mapped to by something in B1.

Everything in B1 lies in some ball centered at 0 in B1. Just take N bigger than the norm of fixed guy, and then its image will be contained in this. And we're taking closure just so that we can write B2 as a union of closed sets, so that we can now apply Baire's theorem.

So this implies by Baire that there exists a natural number n0 such that the image of this ball contains an interior point or it contains an open ball. Now, here's the thing-- T is linear so the image by T of the ball of radius n0 centered at 0 is the same as-- so what I'm saying is, I'm using a little bit of handy notation which I didn't introduce yet, but if I have a subset of a vector space and I have some number outside of it, what I mean by this is the set obtained by taking this set and multiplying it all by the scalar, which is meaningful for vector space.

Now, by scaling, you can check that this thing under the closure is the same as n0 times T of the image of the open ball centered at 1. And then the closure's also with respect to scaling. So N0 times the closure of the image of B 0, 1 contains an open ball so let me just draw, again, a picture to convince you.

So now I have-- let's draw it like this-- this is n0. Now, if I take this set and now multiply it by 1 over n0, I just get the rescaled guy, which is, to use a fancy word, homeomorphic to n0 times the image of the open ball. So this is just a picture to back up what I'm about to say next, which you can also verify not just by pictures but by going through definitions, which implies that T B0 contains an open ball.

So what's this mean? This means there exists some point B2 and a number r positive such that the ball of radius 4r-- so 4 just because it's going to come in handy later with arithmetic, so just by choosing r small-- is contained in the closure of the image of the open ball of radius 1.

So this ball is contained in the closure and therefore, this is contained in the closure. And therefore, there exists a point v1 which is equal to T of u1, which is in T. So v0 is in the closure, so I can find points from the image of this ball close to it. So that's just what I'm writing here.

So there exists a v1 which is the image of some guy u1 in the open ball such that it's close to v0. And how close, so I don't mess this up? 2r. So why did I do that arithmetic?

Simply because if I look at the ball of radius 2r centered at v1-- so v1 is within distance 2r to v0, so everything in here is going to be within distance 4r to v0. And since the ball of radius 4r is contained in here, I obtain-- so this is what I just said a minute ago-- this is contained in the ball of the image of the ball of radius 1, the closure of that.

So now, what I'm going to show is that-- I almost have what I want to prove-- I'm going to show that the closure contains an open ball. That's not quite what I want to show because remember, I want to show that the image of the ball of radius 1 contains an open ball. But what I'm about to show is that the closure of that contains an open ball.

Now, if v is less than r, then if I look at this element 1/2 times 2v plus v1, this is in what? This is in 1/2-- so 2v plus v1, 2v has a norm less than 2r. So this element is in the ball the ball centered at v1 of radius 2r. So that's contained in the closure.

So multiplying by 1/2, I get this is closure, which, as I've said, using the linearity of T and the homogeneity of the norm tells me that this element is in the closure of the ball of the image of the ball of radius 1/2. So this is not crazy. This is an element of this, so 1/2 this ought to be an element of 1/2 times this set. And then the 1/2 comes all the way through because T is linear and the norm is homogeneous, so the 1/2 can come through. But as you're seeing this, think it through slowly.

So that implies that v, which I can write as minus T of u1 over 2. u1 here was just defined as an element of the ball of radius 1, which gave me v1, which was close to v0. So let me keep up this string of inequalities-- plus 1/2 times 2v plus v1.

Now, this is an element of-- don't confuse what I'm about to write down with the notation we used when we were talking about quotients. That's not what I mean here. I mean take this set and add this fixed element to it. So this set here is the set of all elements of the form something in here plus this fixed element.

And now again, by the linearity of T this set here is exactly the same as T of minus u1 over 2 plus ball of radius 0 centered at 1/2, with the closure over all of that. And the closure respects everything that we're doing here.

Now, u1 has radius or has norm less than 1. And therefore, u1 over 2 has norm less than 1/2. Everything in here has norm less thar 1/2. So something with norm less than 1/2 plus something with norm less than 1/2 is something with norm less than 1.

So this is contained in the ball of radius 1 centered at 0, and then closure, and take the image of T. So here, I'm using that this is contained in the ball of radius 1. So that's almost what we wanted to show.

What we have shown is that the ball of radius r is contained in the closure of the ball of radius 1, the closure of the image of the ball of radius 1. Now, again, by scaling this, let me just-- now, the ball of radius r is-- so let me just write down what I'm going to write down. Then I'll explain it.

And therefore, the ball of radius 2 to the minus n times r, which is equal to the set of elements of the form 2 to the minus n times the ball of radius-- elements in the ball of radius r is contained in 2 to the minus n, which equals. So all I did right here was say, again, by homogeneity of the norm, the ball of radius r or the ball of radius 2 to the minus n times r is the same as 2 to the minus n times all elements of the ball of radius r. And that's contained in this by what we've already proven.

So this is contained in the set of all elements of the form 2 to the minus n times things from this set. And again, by homogeneity, 2 to the minus n comes through here and then through here for all n in natural numbers. So let me put-- what I've proven is this and that. So that's almost what I wanted to prove right here, just I want to be able to drop the closure.

And now what I'm going to prove is that I can drop the closure if I take the ball of radius r over 2. So now, we'll prove that the ball of radius r over 2 is, in fact, contained in the image of the unit ball by T. Now, we don't have that immediately because we have the ball of radius 1/2 r is contained in the closure of the image of T of this. There's nothing that doesn't say that the closure of the image of this thing could end up being-- there's nothing that says that that has to be contained strictly within this ball.

So let's show this. Let v have norm less than r over 2, so it's in this ball. Then, as we've proven up here, v is in the image of by T of the ball of radius 1/2.

And therefore, there exists a b1 in the ball of 0, radius 1/2 such that the norm of v minus T of b1 is less than, let's say, r over 4. And now we're going to iterate this. So now, think of this element as lying in the ball of radius r over 4.

And therefore, now I take n equals 2 in here. Thus, v minus T b1 is in the closure of the image of the ball of radius of 1/4, which implies there exists a v2 in the ball of radius 4 such that v minus T b1 minus T b2 norm is less than half of what we had before, so r over 8.

But now you see the game that we're playing. This is now in the ball of radius r over 8, which is implies this is contained in the closure of the image of the ball centered at 0 of radius 1 over 8. And therefore, I can find a b3 in the ball of radius 1 over 8 so that v minus T b1 minus T b2 minus T b3 is less than r over 16.

And then, we just continue inductively. So I said I would do the proper induction argument once and I did it last time. So I will never do it again.

Continuing, we obtained a sequence b k of elements in B1 such that the norm of b k is less than 2 to the minus k. And so there's that. And if I look at v minus sum from k equals 1 to n T of b k, this is less than 2 to the minus n minus 1 times r. Let me make sure my-- yeah.

Now, we've used the fact that B2 is complete. We haven't used the fact that B1 is complete yet. So we ought to use it at some point.

We're going to use it now by showing that v has to be, in fact, inside the image of the radius of the ball of radius 1. So these b k's form a Cauchy sequence-- or I shouldn't say they form a Cauchy sequence, but their sum forms a Cauchy sequence. Let me write it this way.

The series is absolutely summable because the norm is bounded by 2 to the minus k, which is summable. And since B1 is a Banach space, the series must be summable, which implies there exist a B and B1 such that B is equal to the sum k equals 1 to infinity of b k. Moreover, the norm of b, which is equal to the limit as n goes to infinity of the norm of n k equals 1 b k, this is, by the triangle inequality, less than or equal to the limit as n goes to infinity of-- equals 1 to n b k, which is less than-- well, is equal to-- sum from k equals 1 to infinity of the norm of b k.

And each of these is less than 2 to the minus k. So k equals 1 to infinity of 2 to the minus k. And that sum is just 1. So the norm of b is less than 1 using the first property of this sequence.

The second property of this sequence then shows that v is equal to the image of b by T. Moreover, since T is continuous, T b, which is equal to the limit as n goes to infinity of t applied to k equals 1 to n b k, which equals, by linearity, limit as n goes to infinity of sum from k equals 1 to n T of b k. By the second property, the limit as n goes to infinity of T of b sub k is equal to v.

And therefore, B is in the image of the ball of radius 1 by T. And thus, the ball of radius r over 2 is contained in the image of the ball of radius 1. So that's the special-ish case that I wanted to prove in terms of open sets.

So what I've shown is that, if you like, the interior point 0 remains an interior point. Now let's show that implies the full claim of what I want to prove for the open mapping theorem, that every open set gets mapped to an open set in B2. We'll just use translation.

So in scaling again, suppose u subset of B1 is open and B2 is the image of something in u, so this is in T of u. Then there exists an epsilon positive such that B1-- so remember, u is open-- so B1 plus all elements of the ball centered at 0 of radius epsilon and this, which is equal to the ball centered at B1 of radius epsilon is contained in u since it was open.

Now, since there exists a delta positive such that ball of radius delta is contained in the ball of radius 0, 1, this implies that the ball of radius-- again, by homogeneity, the ball of radius epsilon times delta is contained in the image of the ball of radius epsilon, again, because T is linear and the norm is homogeneous. So let's go back to how we had it written down.

So b2 plus-- so this is the of epsilon times delta. I got my delta and epsilons backwards. Anyways, this is equal to b2 plus epsilon times the ball of radius delta. So this is actually equal to the b2 plus the ball of radius 0 centered at 0 of radius epsilon times delta. The epsilon I can pull out.

Now, this is contained in b2 plus epsilon times the ball of radius 0, 1. Yeah, I don't know why I wrote this down. I don't think I needed it. Anyways, OK.

And this is equal to T of b1 plus epsilon T of B 0, 1. And again, by linearity and homogeneity, this is equal to the image of b1 plus epsilon ball of radius 1, which I can say is b1 plus ball of radius epsilon. Now, this epsilon, remember, was chosen so that this ball here is contained within u. And therefore, this is contained within u and the image is, therefore, contained in the image of u.

Yes, so I'm not sure why I decided to write this down. This is what threw me for a loop. But anyways, so this is the point-- that we can just take the special case and shift things around to get the general statement for open sets getting mapped to open sets.

So from the open mapping theorem-- I don't know, it seems almost topological-- but we get what's called the closed graph theorem, which gives you sufficient conditions to be able to check if something is continuous. It's a little bit more convenient. And I'll explain why in just a second.

So this is the closed graph theorem. But first, I need to state just a simple theorem before I actually state the closed graph theorem. If B1 and B2 are Banach spaces, then their Cartesian product, which I can give a natural vector space structure on from B1 and B2, just the sum of an ordered pair of elements is just the entry by entry sum.

But I can also put a norm on it coming from these two, with norm-- so for an ordered pair b1 and b2, the norm of this is just defined to be the sum of the norms in the respective spaces. So this is a norm space. But moreover, if they're both Banach spaces, this is a Banach space. So it's not difficult to prove just based on the definition.

I'm not going to write the proof. I will leave it to you, the proof. Again, it's not difficult. Simply from how we've defined the norm a Cauchy sequence in B1 cross B2, the first entry will form a Cauchy sequence because of this definition of norm, and the second entry of the sequence will form a Cauchy sequence in B2.

Both of those have limits. So then you can prove the sequence consisting of ordered pairs has a limit. It's the same way you can prove that R2 is complete assuming R1 is complete.

All right, so now, I can state the closed graph theorem, which is the following-- if B1, B2 are Banach spaces and you have a linear operator from B1 to B2-- so all you know is it's linear, don't know that it's bounded-- then there's an equivalent condition you can check to see if it is, in fact, a bounded linear operator. Then T is a bounded linear operator from B1 to B2 if and only if the graph of T which is defined to be the set-- let's see, what notation do I use-- u, v such that v equals T u-- let me write it this way-- which is a subset of B1 cross B2, is closed.

So a linear operator is a bounded linear operator if and only if the graph of this linear operator given by u, T u is closed. Now, why is this a little bit easier or I say convenient than just checking if something's a bounded linear operator? Well, maybe it's difficult to prove the boundedness property that we have that's equivalent to continuity.

So a bounded linear operator is a linear operator which is continuous. So maybe it's difficult to prove the bound. So you have to go back and try and prove-- just for a God-given or instructor-given operator-- then you try and go back and prove continuity. And continuity says, well, take a sequence u n's converging to u. You then have to prove that T of u n converges to T of u.

But there's kind of two statements in there. You have to prove T of u N converges and that limit is equal to T u. What the closed graph theorem does it eliminates one of those steps. Because to prove that the-- let's think about that, that the graph is closed in B1 cross B2.

What does that mean? That means you have to check that it's closed under taking limits of sequences. So you have to show that given a sequence u n tending to u and T of u n converging to v, that v is equal to T u.

So you get to assume already that T of u n converges. You just now have to check that the thing it converges to is actually equal to the image of the limit of the u n's. I hope that makes sense and explains why I said this is actually a little more convenient than continuity, or at least useful.

So this is a two-way street. There's always, typically, one side of the street that's plowed. So let's do this direction, assuming that T is a bounded linear operator, and show that the graph is closed. So if T is a bounded linear operator-- let me just write "suppose" and then start a new sentence.

So let u n T of u n be a sequence in the graph of T such that u n converges to u and T of u n converges to v. To show the graph is closed, we have to show that that the pair u, v is in the graph, that v is equal to T of u. But this follows from continuity.

Then v is equal to the limit as n goes to infinity of T of u n. And since T is continuous, this is equal to [INAUDIBLE] u n, which equals T u. Thus, the ordered pair u, v is in the graph.

So let's now prove the opposite direction. It's still not that difficult. So what I'm first going to do is I'm going to draw a diagram. This may be the only diagram I ever draw in this-- a commuting diagram-- maybe the only one I draw in this class. We'll see.

So T takes B1 to B2. I have the graph of T. And now, I'm going to define two maps going from the graph to B1 and from the graph to B2.

The graph sits as a subset of B1 cross B2. So this first map going from the graph is just going to be the projection onto the first entry. And pi 1 and then pi 2 will be the projection onto pi 2.

Now, to finish this graph, I need to have an arrow going from B1 up to gamma of T. So I just first want to note-- actually before I note it, I'm going to go ahead and draw the arrow. And then I'll tell you what S is.

So pi 1 with respect to the graph of T-- this is a surjective map. So let me actually define these things. A graph of T to B1 via pi 1 of an element of the form u, T u equals u, and then pi 2 from the graph of T to B2 is just take the second element, u T u equals T u.

So my point here-- so first note-- gamma of T is a Banach space. Why? Gamma of t is a subspace of B1 cross B2 because T is linear and it's closed. So a closed subspace of a Banach space is, again, a Banach space, since it's a closed subspace of the Banach space B1 cross B2 with this norm that I defined in the earlier theorem.

Now, pi 1 and pi 2-- these are both continuous, viewed now as maps from the Banach space of the graph to B1 and B2. Pi 1 is a bounded linear operator from the graph T, B1. And pi 2 is a bounded linear operator to B2.

Why is this? I mean, this is pretty clear. So the graph of the norm of things in here are just-- so this is because if I take the norm of pi of-- let me write it u, v now, where v is standing for T u-- this is equal to v, which is less than or equal to-- so this is pi 2 of this one-- which is less than or equal to, and then the same thing with pi 1.

So pi 1 and pi 2 going from the graph to B1 and B2-- these are bounded linear operators. And pi 1, when restricted to the graph is, in fact, bijective. It's one to one and onto. So I've used "moreover" again. So let me just write moreover more over.

Pi one going from the graph to B1 is one to one and onto, bijective. Everything in B1 gets mapped to by pi 1 from the graph. If you have something u1 here, then its image is u1 T of u1. That's a unique element in the graph, since T is a function. There's only one element in the graph corresponding to a given u.

And so let's pause the proof here because I forgot to write a corollary after the open mapping theorem. In fact, let's state it over here. So this was the end of the proof of the open mapping theorem. So there's space for the corollary, which I wanted to write here.

It's the following-- if B1 B2 are Banach spaces, T is a bounded linear operator from B1 to B2, which is bijective, meaning one to one and onto, so it has an inverse, then T inverse is a bounded linear operator from B2 to B1. So if I have a bounded linear operator from one Banach space to another and it's bijective, its inverse is automatically continuous. And the proof of that just follows from the open mapping theorem.

So I'm going to write it in one line because that's all the space I'm going to give myself. T inverse is continuous if and only if, for all u that's open, for all open sets u and v1 which is in the image, the inverse image of T inverse-- or I should say the inverse image of u by T inverse, which you can just check is equal to T of u-- is open. And that's true by the open mapping theorem since a bijective map is surjective.

So every bijective bounded linear operator automatically has a bounded inverse. It's also linear. I mean, I didn't say that, but if I have a linear operator which is bijective, then its inverse is also linear.

So now coming back to this proof of the closed graph theorem. So we have the graph, which is a Banach space in its own right as a subset of B1 cross B2, as a closed subset of B1 cross B2. I have pi 1 and pi 2, which are bounded linear operators between B1 and B2. I didn't say that they're linear, but that should be clear.

And pi 1, when restricted to the graph, is one to one and onto. Pi 1 here just takes the first element of what's in the graph and spits out-- let me, instead of having T u there, let me have v like that. We just know that v is equal to T u.

And so this is one to one and onto. It's bijective. Thus, it has an inverse which is a bounded linear operator, by the corollary that I stated over there.

It is a bounded linear operator, which implies that T, which I can write as-- I shouldn't have S inverse, defined to be the inverse. And therefore, T, which is equal to S of pi 1 going from B1 to B2 is now the product of two bounded linear operators, pi 1 restricted from B1 from B1 up to the graph. And then S-- no, no, no. I'm messing this all up. Hold on-- S pi 2.

OK, now this makes sense. So S, which is the inverse of pi 1, is a bounded linear operator. Pi 2 is a bounded linear operator. And therefore, their composition is also a bounded linear operator, which implies that T is a bounded linear operator. So I made kind of a mess of that having to go back and forth, but the proof is simple enough. I'm sure Professor Melrose would have just drawn the picture, but I decided to make a mess of it.

So those are some pretty important theorems that follow from the Baire category theorem. So we got uniform boundedness from Baire category. We got open mapping from Baire category.

We got closed graph from open mapping. If you're a lover of logic, think about it a little bit-- open mapping implies closed graph, but you can also show that closed graph implies open mapping. So as logical statements, they're equivalent.

Now, we're going to move on to the Hahn Banach theorem. So I haven't done many examples here going into this kind of general theory, but don't worry. There will be plenty of examples in the assignments of using these theorems, and so on.

So the Hahn Banach theorem-- these theorems before were all kind of answering a question. Maybe I didn't state the questions as clearly. Closed graph is kind of-- well, it doesn't so much answer question as give us an alternative to proving continuity. Open mapping you can think of as trying to answer this question-- if I have a bijective bounded linear operator, is its inverse a bounded linear operator? And uniform boundedness is the answer to the question for at least a sequence of bounded linear operators, does pointwise convergence imply, or pointwise boundedness imply uniform boundedness?

Now, the question that the Hahn Banach theorem tries to answer is the following-- given the general non-trivial normed norm space V, is the dual space given by simply the zero vector? So at the end of last class, I defined the dual space. Recall, this is equal to the bounded linear operators from the norm space V to the field of scalars which is a Banach space because it's a space of bounded linear operators from a norm space to a Banach space, so it's a Banach space.

And we usually refer to elements of the dual as-- I'm not sure if I said this last time, but we don't refer to them as bounded linear operators from the vector space to the field of scalars, but as functionals. Because the classical space is where function spaces-- the classical Banach spaces were spaces of function. So the things that ate them and spat out a number were called functionals, so evolving from function of functions and functions of lines.

So the question is, if I have just the norm space, is the dual space kind of nontrivial in general? So last time, I hinted that for certain spaces, you can actually write down the dual space explicitly. You can at least identify the dual space in an explicit way.

I hinted last time that the dual of little l p prime is, in fact, equal to l p prime, where p prime is defined as the dual exponent. So 1 over p plus 1 over p prime equals 1. And this is for p bigger than or equal to 1 and less than infinity, but not for p equals infinity.

And if you remember last time, C0, which is a set of sequences which converts to 0-- I can't remember if I wrote this down at the end of last class, but you can also identify its dual space with little l1. So just to give you some examples of spaces that do have nontrivial dual space-- examples of norm spaces which have nontrivial dual space are given by the little l p spaces. But now the question is, in general for a norm space, is the space of functionals, is the dual space nontrivial?

And this is a statement of the Hahn Banach theorem that there's, in fact, a lot of elements in the dual space. Now, we're not going to get to the statement or proof of that in this lecture. Because we first need to at least go over, or state, or you can say result, axiom from set theory that we'll need.

So first, like I said, we need an axiom or recall an axiom, a certain lemma, from set theory. So first, let me set down the appropriate terminology. So partial order on a set E is a relation, meaning just a subset of E cross E, denoted by this-- but we usually don't identify it as a subset of E cross E-- such that three things occur.

So you think of it kind of as a less than or equal to. For all E in E, E is related to E. I will say less than or equal to, even though this may have nothing to do with less than or equal to.

For all e and f in capital E, E less than or equal to f and f less than or equal to e, these two assumptions imply that e is the same element as f. And transitivity-- so this is reflexivity. I'm not sure what would you call this one. I can't exactly remember.

For all e, f, g in E, the two assumptions E less than or equal to f and f less than or equal to g implies e less than or equal to g. So this is the definition of a partial order. So to go with this, we say an upper bound of a set D contained in E is an element e in E such that for all d in capital D, d is less than or equal to e.

And a maximal element of the set E is an element e in E, which nothing lies bigger than it, essentially. Nothing maximizes it or majorizes it, such that if f is in E and e is less than or equal to f, then e is equal to f. And a similar definition for a minimal element. So this was the definition of a maximal element. And this is the definition for a minimal element.

Now, note the maximum element may not sit above everything in E necessarily. It may just be kind of off to the side of everything in E. Because this doesn't assert that you can always check to see if between two elements, if one is bigger than the other.

That's something a little more restrictive, which is the following-- if E and less than or equal to is a partially ordered set, meaning a set with a partial order, a chain in E-- so maybe a better way to say this is a set C is a chain if, but a chain in E is a set C such that for all e, f in C, either e is less than or equal to f or f is less than or equal to e.

So a chain is something so that you can, for any two elements in the set, compare whether one is bigger than the other. That's what a chain is. But for a general partial order, it doesn't necessarily need to be the case that you can always check to see if one is bigger than the other.

For example, your partial order could be on the power set of some set. And the partial order is inclusion, whether one subset is a contained in another. Then it satisfies these three properties, but there's sets that cannot be compared to each other.

So let me write this as lemma due to Zorn. We're not going to prove this. Even though I'm writing "lemma," just take it as an axiom, an axiom of Fraenkel set theory that goes with it, which is the following-- that if every chain in a nonempty-- of course, we're considering nontrivial stuff, so in a nonempty-- partially ordered set, E has an upper bound, then E has a maximal element.

So if you can check that every chain has an upper bound, then you get to conclude that the partially ordered set has a maximal element. Now, we'll give a simple application of this at the start of the next lecture. But first, let me put this into your brains to marinate on.

So we're going to use Zorn's Lemma to prove the Hahn Banach theorem. And I'll go into why that is next time. But you can use Zorn's Lemma to prove other things, and it is used to prove other things.

And first off, from Zorn's Lemma you can, in fact, prove the axiom of choice, which says given any collection of sets, you can essentially choose an element from each set stated in a very precise way. Another way, which we're going to use at the beginning of next time, is to prove the following-- but first let me make a definition. A Hamel basis, H which is a subset-- not a subspace, but a subset-- of V, a vector space, is a linearly independent set such that every element of v is a finite linear combination of elements of H.

So from linear algebra, for finite dimensions, this is in some sense how one can define the dimension is you find a basis and then the cardinality of that basis is always the same. So a Hamel basis for R n is just the vectors with 1 in one of the entries and 0 otherwise. So for R 2, it's just 1, 0, and 0, 1. A Hamel basis for, let's say, little l1 would be one in the first, followed by 0's, 1 in the second, 0 elsewhere, 1 in the third, 0 elsewhere-- the set of these elements.

Now, the question is, does every vector space have a Hamel basis? And using, next time, Zorn's Lemma, we'll show that indeed, every vector space has a Hamel basis. And in fact, that Hamel basis can be quite big. But that'll be a simple application we do next time, is that via Zorn, you can show that every vector space has a Hamel basis. We'll stop there.