[SQUEAKING] [RUSTLING] [CLICKING]

PROFESSOR: All right, so let's continue our discussion of Banach spaces. So let V be a norm space, meaning a vector space with a norm on it. And last time, a Banach space was defined to be a norm space such that the metric induced by this norm is complete-- all Cauchy sequences converge. So if you want to check that a norm space is a Banach space, you have to take a Cauchy sequence and show that it converges in the space, which we did last time for the space of bounded continuous functions on a metric space.

Now, there's an alternative, useful way of checking to see if a space is a Banach space, which we'll use in a minute. But to state it, I need to introduce a definition real quick. So let vn be a sequence in V. And I'm going to abuse notation and write subset of V, even though that's a sequence. I'm using this notation, though. So this means let vn an a sequence in V.

The series, which is just right now an expression, just chalk on a board right now, it's just a symbol-- we say this series is summable if the sequence of partial sums, which are now elements in the norm space-- so if this sequence of partial sums converges. We say that the series vn is absolutely summable if the series of involving now these non-negative numbers converges.

OK, so this is just like the definition of the convergence of a series of real numbers, which you dealt with in earlier analysis and absolute convergence, only here I'm using the terminology absolutely summable because this is the terminology Richard Melrose used, so I want to stick to what he's using. He's Australian, so I think maybe that has something to do with it. So that's one of the unfortunate things about this, is I can't tell if you're laughing, but I'm going to assume you're laughing.

OK, so absolutely sum of all means the sum of the absolute-- of the norms converges. And you have this theorem, just like from real analysis, which you saw at one point, that if vn-- so if this series is summable-- is-- I'm missing an adjective there. If I have an absolutely summable series, then the sequence of partial sums-- this is a Cauchy sequence in the space V. So again, we're working through in a normal space V.

All right, and the proof is the same as in the real numbers case. So proof I'll leave to you. This is just a simple exercise. And it's the same as for V equals R.

Now, notice I said something which is strictly weaker than what you encounter in either case V equals R. In the case V equals R, you have the theorem that if I have an absolutely summable series, then it's summable. Every absolutely convergent series is convergent.

But I didn't say that here. I just said that the sequence of partial sums is Cauchy, not necessarily convergent. So when is the sequence of partial series convergent? When can I say that an absolutely summable series in this norm space is summable? And I can say that precisely when it's a Banach space. So the theorem that we're going to prove is that V is a Banach space if and only if every absolutely summable series is summable.

OK, so this characterizes Banach spaces as those spaces for which this theorem you have from real analysis, that every absolutely convergent series converges, is precisely that. Every absolutely summable series is summable. And sometimes, that is an easier property to verify than going through the whole Cauchy business. And sometimes, it's exactly the same amount of work. We'll use this later when we deal with integration and measure theory to prove that the big Lp spaces are Banach spaces.

All right, so we have two directions to prove-- that if V is a Banach space, then we get every absolutely summable series is summable. And this is pretty straightforward. So one direction-- if V-- suppose V is a Banach space. Then if vn, if this series, is absolutely summable by the previous theorem, which I didn't prove, but is very easy to prove, I get that the sequence of partial sums-- this is a Cauchy sequence in V. And because V is a Banach space, every Cauchy sequence converges. And therefore, it converges in V. And therefore, the series is summable.

So that direction simple enough. Let's go the opposite direction and show that every-- that the condition that every absolutely summable series is summable implies that V is a Banach space. So every absolutely summable series is summable.

Now, we want to show that every Cauchy sequence converges in V. So let vn be a Cauchy in V. So what we're going to do is we're, in fact, going to show that there's a subsequence of the sequence that converges. So what we're going to show-- so that this sequence has a convergent subsequence.

And once we've done that, we're done because remember back to your real analysis days. If a Cauchy sequence has a convergent subsequence, then the entire sequence converges. And vn converges by metric space theory, all right? So real analysis stuff.

OK, so let's find this subsequence. And basically, we're going to build this subsequence up by speeding up the convergence of vn or, if you like, speeding up the Cauchiness of vn. So the fact that the sequence is Cauchy implies that for all natural numbers, there exists a natural number k, also a natural number, such that for all nm bigger than or equal to N sub k, we have that the norm of v sub n minus the norm of v sub m is less than 2 to the minus k.

All right, why did I choose to the minus k? Because that's summable, all right? And you'll see. And so what we're going to do is build up essentially a telescoping sum from these-- from Well-chosen guys. So define n sub k. What is this going to be? This is going to be equal to N sub 1 plus plus N sub k.

So n sub 1 is less than n sub 2 is less than n sub 3 because at each stage, I'm adding a natural number. So n sub 1 is equal to capital N sub 1. n sub 2 is equal to a capital N sub 1 plus n sub 2. These are natural numbers. So I'm always getting bigger at each stage. So this is an increasing sequence of integers. And for all k, n sub k is less than or equal to N sub k, capital N sub k because little n sub k is equal to sum integers plus capital N sub k.

And so the v sub n sub k's are going to be essentially the guys which converge. Thus, for all k natural numbers, I get that v sub n sub k plus 1 minus v sub n sub k-- if I take the norm of that, so n sub k is bigger than or equal to capital N sub k. n sub k plus 1 is bigger than or equal to n sub k, which is bigger than or equal to capital N sub k. And therefore, by this condition-- how n sub k's are chosen-- so this, this, this, and what's in blue tells me that this is going to be less than 2 to the minus k.

And therefore, thus, the sum v sub n sub k plus 1 minus v sub n sub k-- this is absolutely summable, right? Because the norm of this is less than 2 to the minus k, which you can sum. And by our assumption, that every absolutely summable series is summable, this implies that-- I didn't even say anything after that-- is absolutely summable, which implies that this is summable, which implies the limit as m-- or let me instead just finish with that-- i.e. the sequence of partial sums k equals 1 to m v sub n sub k plus 1 minus v sub n sub k-- this sequence of partial sums converges in V.

OK, so to recap again, we started off with a Cauchy sequence. We go out in the Cauchy sequence far enough and pick certain guys so that they're pretty close to each other. And how close? So close that the sum of their norms is finite, so that it's absolutely summable. And therefore, the series is summable by our assumption.

But this is essentially a telescoping sum. Thus, the series v sub n sub m, which equals-- so the sequence v sub n sub m, which is the sum from n equals 1 to m of v sub n sub k plus 1-- let's see, let me put a minus 1 here, minus v sub n sub k plus v sub n sub, m equals 1 to infinity converges in V. And I'm done.

So v sub n sub m is equal to this telescoping sum plus-- so when I add this up, terms cancel. And I just pick up the last one, which is when I pick up m sub 1 here minus the first one, which is v sub n sub 1 here. So if I add on v sub n sub 1, I just pick up v sub n sub m.

Now, as m goes to infinity, this converges to something because this sequence converges. And this is just fixed in m. So this sum converges. And therefore, v sub n sub m converges. And thus, this subsequence of our original Cauchy sequence converges, proving that the Cauchy sequence converges in V. And we're done.

OK, so Banach spaces, these are a nice generalization of the spaces that you worked with in real analysis and linear algebra-- Rn, Cn, and so on. So what are the analogs of matrices, which you had to use in calculus and linear algebra? This is going to lead to our next topic, which is operators and functionals-- so operators being the analog of matrices that take one vector into another vector, functionals are the analog of taking a vector and taking its dot product with a fixed vector, spitting out a real number. So functionals will eat vectors and spit out real or complex numbers, depending on the field that you're working in.

So let me write down just an example to keep in mind, as far as operators go. So I want you to keep this example in mind, which was the whole reason for really a lot of building all this machinery. I mean, this example came first, and then the machinery came-- was built later to be able to say all we can about these kind of operators-- or these kind of transformations.

And depending on if you're in my class or not, maybe you saw a question about such a creature on an assignment or maybe on an exam. So let K be a function on 0, 1 cross 0, 1, let's say, into the complex numbers. And let's assume it's continuous for f, a continuous function on 0, 1. We can define a new function from f, Tf of x, to be the integral from 0 to 1 of K of xy times f of y dy.

Now, these are things that-- I'm about to write a few things that you can just check by hand. But you can then check then Tf is also a continuous function. And it's linear in the argument f. And for all lambda 1, lambda 2 in C, f1, f2 in 0, 1, T lambda 1, f1 plus lambda 2 f2 equals lambda 1 Tf1 plus lambda 2 Tf2.

And it has another property, which I'm going to say in a minute, namely that it's continuous on the space of continuous functions. So we've already proven that this space of continuous functions on 0, 1, this is a Banach space, right? This was a special example of the space of bounded continuous functions on a metric space we considered before because on the closed and bounded interval 0, 1, every continuous function is bounded. So this is just equal to C subscript infinity of 0, 1. So we know this is a Banach space.

And so this is an example of what's called a linear operator. So definition-- let V and W be vector spaces. So you should have seen a linear transformation. I'm going to call it a linear operator. I'm just recalling what it means to be linear.

Let V and W be vector spaces. You say a map T from V to W is linear-- so linear, if for all lambda 1, lambda 2 in your field of scalars-- if they're either R or C-- and for all v1, v2 in V, T of lambda 1 v1 plus lambda 2 v2 equals lambda 1 T v1 plus lambda 2 T v2.

So in a map from one norm space-- so this is just a note off to the side. This is how I'm going to be using-- well, I'm not even going to write it down. So given two norm spaces, a linear map between them I will most often refer to as a linear operator. Rather than linear transformation, which is what you probably heard in linear algebra, I'll refer to these as operators.

Something I meant to say as well was why do we care about such a guy like this other than it looks nice? You care about guys like these because operators of this form are essentially the inverse operators of differential operators. I mean, you know that from the fundamental theorem of calculus. The inverse operation of taking a derivative is taking is integrating.

So it shouldn't be a surprise that the inverse operator, meaning if I take f as my data to some ODE, can be written as this, as this kind of linear operator. That shouldn't come as too much of a surprise. So that's why we care about them, is that operators of this form arise as the inverses of taking differentiable operators.

OK, so now, in this class at least, we're not just interested in any old linear operator. We're going to be interested in a certain class of linear operators, those-- so those which are continuous. So let me recall for you an equivalent way of saying that a map is continuous.

So this is just any map, any function, not necessarily a linear operator is continuous on V if-- and there's two ways to say this-- for every sequence vn converging to V-- so let me write it this way. For all v in E for all sequences converging to V, we obtain that T of v sub n convergence to T of v.

And an equivalent way of stating this in terms of what one would call a topological notions is that the inverse image of open sets are open. So for all open u in W, the inverse image of u, which I will recall for you is the set of v in capital V such that T v is in u-- I'm not saying that T is invertible. That's the inverse image. The set is open in V. Remember, the notion of an open set is that for every point in that set, there's a small ball that's contained entirely in the set centered at that point.

OK, now, for linear maps, there's a very simple way or equivalent way of finding when it's continuous on a norm space. Now, on finite dimensional spaces, any linear transformation-- every linear transformation is continuous. I should say that. So if you take any linear operator from Rn to Rm Cn to Cm or Rn to Cm, anything between two finite dimensional spaces, it will always be continuous if it's linear. That is not always the case between two Banach spaces.

Now, again, is there a more efficient way of checking when something is continuous? Or what's an equivalent way of saying that? Or a more useful way is the following characterization that we have.

So a linear operator T between two norm spaces now-- so between two norm spaces is continuous if there exists a C positive such that for all v in capital V, if I take T of v and take it's norm in W, this is less than or equal to a constant times the norm of v in capital V.

OK, now, in this case, we say-- instead of saying that T is a continuous linear operator, we say T is a bounded linear operator. So we don't really say continuous linear operator. We say bounded linear operator.

Now, this doesn't mean the image of V is a bounded set in W. That's not what this means. The only linear operator that takes the bounded set-- that takes a vector space into a bounded set is the zero operator. So we're not saying that it's taking all of V to a bounded set. But what this inequality does say is that it takes bounded subsets of V to bounded subsets of W.

So let's prove-- let me put a star by this condition, so I don't have to write it out so much. OK, so let's go in this direction. So let's assume star and prove that this linear operator T is continuous. And this is not too difficult to do.

We'll use this first-- we'll use this first characterization of continuity. Let v be in V. And suppose v sub n is a sequence in V converging to v. Then by star, if I look at T sub v sub n minus T sub v norm in W, this is less than or equal to a constant times the norm of v sub n minus-- well, first off, I'm kind of using-- let me add one little step. I'm using that it's a linear operator, meaning I can write this as T of v minus v sub n.

And therefore, this is less than or equal to a constant times v sub n minus v in capital V. And so the norm of T of v sub n minus T of v is less than or equal to some fixed constant depending only on T times v sub n minus v. This goes to 0 as n goes to infinity. And therefore, by the squeeze theorem, this thing on the left must go to 0. It's always trivially bigger than or equal to 0, by the squeeze theorem. And therefore, T of v sub n converges to T of v. So that takes care of one direction.

OK, so we've shown that this boundedness property of T implies that T is continuous, that the linear operator T is continuous. Now, let's show that continuity implies this boundedness property. And so for the continuity guy-- so I'm assuming T is continuous-- I'm going to use the second characterization of continuity here.

So then the inverse image of every open set in W is an open set in V. So the inverse image of the ball centered at the zero vector in W of radius 1-- so let me just recall this is a set of v in V such that T v is in the ball. This is an open set in V because the ball of radius 1-- so this is the ball of elements in W such that their distance to 0 is less than 1. This must this is an open set in W. And therefore, its inverse image must be an open set and V.

So we have here 0, 1, everything inside. Don't include the boundary. This must be an open set in V. And T takes one to the other.

Now, what do I know? 0 is in here. And every linear transformation takes 0 to 0. So 0 has to be in the inverse image because any linear map takes 0 to 0. So 0 has to be in the inverse image.

And therefore, since this set is open-- so this is 0. Since the set is open, I can find a small ball of radius r centered at 0, which remains inside the set V. And therefore, it gets sent to some other set, which is a subset of W. So this is the picture. Let me write down the math that goes with it.

Since T of 0 equals 0, that implies 0 is in T inverse of-- which implies since this is open, there exists an r positive such that the ball in V centered at 0 of radius r is contained in T inverse of-- the inverse image of the ball of radius 1 centered at 0 in W. Just look at the picture here.

Let v be in V. And let's not look at 0 because v equals 0 will satisfy this inequality no matter what C you have. So we just need to look at v not equal to 0. I claim that I can take the constant to be 2 over R, all right?

OK, then if I take v and I rescale it-- well, let me not write it as dividing v. Let me write it this way, r over 2 length v v v times v. So this is a vector in capital V. What is its length? Its length is r/2. So if I take it's length in v, this is equal to r/2 is less than r, which implies that r over 2 over v v is in the ball of radius r in V centered at 0. And therefore, so it has to be in this blue disk. And therefore, it gets mapped to something in this blue guy here, which is contained inside-- remember, this big yellow thing was the ball of radius 1 centered at 0-- of radius 1 in W.

And therefore, the length of T r over 2 v v v W must be less than 1. And now, scalars pull out of linear transformation. So this comes out. And then it comes out of the norm by the homogeneity of the norm. And so I can divide through by-- multiply through by 2 norm v over r. And I get that T of v norm W is less than 2 over r norm V. And therefore, star holds with C given by 2 over r.

OK, so continuous linear operators between norm spaces, we call them bounded linear operators because they satisfy this boundedness property, namely that they take bounded sets to bounded sets. Now, it's going to become quite tedious for me to keep writing norm sub W, norm sub V, and so on. So I'm going to stop using the subscripts. But it should be pretty clear from the context where the norm is.

If I have a linear operator T from V to W, T of V-- so if I have a bounded linear operator or linear operator from V to W and you see me write norm of T v, you should equate this to the norm of T v in W. Or if you see v, you should equate this-- and v as an element of capital V, then you should interpret this as the norm of v in capital V. So I'm dropping subscripts just to save having to write too much. And then it'll soon get tedious.

OK, so the definition-- in fact, before I do that, let's take a look at this linear operator I wrote up there a minute ago. Can we see that that's a bounded linear operator on the space of continuous functions? So given by-- where k is a continuous function-- this is a bounded linear operator.

So it's clear-- pretty clear to see that it's linear in f, right? Scalars pull out and so on. Let's check that it's bounded. So recall that the norm on C 0, 1 is the infinity norm.

So let f be a continuous function. Recall the norm on this space is the infinity norm given by sup of x in 0, 1, f of x, which is, in fact, a maximum. It's attained at a certain value. But I'll just write sup anyways.

And so now, we want to estimate the norm of T of f of x in terms of the norm of f. That's what this boundedness property is. Then for all x in 0, 1, this is equal to the integral of k xy f of y dy. Now, the absolute value of the integral is less than or equal to the integral of the absolute value. So this is times the absolute value of k xy times f of y dy.

Now, f of y for all y in 0, 1 is less than or equal to the infinity norm of f. So that only makes the integral bigger. And the same thing for k, right? k is a continuous function on 0, 1 cross 0, 1. And therefore, it's bounded by its-- it attains a max on this set, being a continuous function. So I can also replace it by its infinity norm.

And here, you should interpret this as being the infinity norm on continuous functions on this set. So this infinity norm is the sup of k of xy for x and y in 0, 1 cross 0, 1. And therefore, this equals-- these are just two numbers. And this held-- holds for all x. And therefore, its supremum is bounded by this number as well, so Tf-- so this is a bounded linear operator with constant given by the infinity norm of k.

This k is usually referred to as a kernel. So if I've said that before and haven't explained where that comes from or why I use that word, this is usually referred to as the kernel of this linear operator. So there's an example for you. I've already used up both boards? Yeah, I missed the big room that had three of these.

Now, given two norm spaces, V and W, we can consider the set of all bounded linear operators from V to W, which we denote by B V, W-- scripty looking B-- set of T. T is a bounded linear operator. So it's not difficult to see, again, that-- or to see that this is a vector space.

So let me put this in a remark, that it's clear that this is a vector space. The sum of two linear transformations is a linear transformation-- or linear operator. The scalar multiple of a linear operator is a linear operator. And then those two operations preserve continuity. So this is clearly a vector space.

Now, we can put a norm on this space. We define the operator norm of an element in here. This is defined to be the supremum over all unit length vectors v of T of v. So I want to recall that this norm is being taken in W because T of v is an element in W. This norm here is being taken in V because little v is an element in V.

So maybe it's not clear at first. But let's go ahead and prove it that, in fact, the operator norm is, in fact, an actual norm so that this becomes a norm space. So the operator norm is an actual norm. It's not just a norm because I said it is. So let's prove this.

It's not-- again, so it's not too difficult to see. So let's prove definiteness, namely the norm of T is 0 if and only if T is 0. So if T is a zero operator, then clearly its norm is 0. Suppose T of v equals 0. So this sup is 0 if and only if T of v is 0 for all unit length v. So suppose T is an operator. So that T of v equals 0 for all unit length v. Then that implies that T of-- that 0 equals T of v equals T.

All I'm saying is you rescale now, OK? 0. So it implies for all v in V take away 0 that 0 equals T. So t is the zero operator if its norm is 0.

2. Homogeneity follows from the homogeneity of the norm on W. So lambda T, this is equal to-- so the norm of lambda times T, this is equal to the sup of v equals 1 of lambda T v. And this is equal to sup equals 1 times lambda times Tv. And if I take a set and multiply it by a non-negative number, then that non-negative number comes out of the sup.

Hopefully, it was one of the first exercises you were ever given on supes. And therefore, this equals-- and now, so that proves homogeneity of the norm. And now, the triangle inequality follows from the triangle inequality for the norm on W again.

So I take two operators and two bounded linear operators. And I take an element of V with norm equal to 1. Then S plus T applied to v norm, this is equal to Sv plus Tv. And by the triangle inequality for the norm on W, this is less than or equal to S times v plus T times v.

And now, S times v-- so again, v's a unit length vector-- is less than or equal to the sup over all those norms, which is, again, just the operator norm of S and then the operator norm of T. So I've proven that for all unit length v, S plus T applied to v in norm is less than or equal to this number here. And therefore, the supremum over all such numbers as v ranges overall unit length vectors, which is the least upper bound of that set, must be-- sit below this number here, which implies that-- OK? And therefore, the operator norm is a norm both in name and in actuality.

So if you like, what we did a minute ago here is we showed that-- so coming back to over here for this bounded linear operator from continuous functions to continuous functions, what this tells you-- so first off, if f has unit length, then I've shown that Tf in L infinity form is less than or equal to the L infinity norm of K. And therefore, I've shown that the operator norm of T, where T is defined over there, is less than or equal to the operator norm for K. We're actually a little wasteful there. This is not equality, actually. But I'll let you think about that when you have free time.

OK, so we've talked about bounded linear operators from one norm space to another. What more can we say about this new space we formed from two norm spaces? When is this space complete, for example? What are sufficient conditions?

And so theorem-- if W is a Banach space-- again, V and W are norm spaces. If I assume W is a Banach space, then the space of bounded linear operators from V to W is a Banach space no matter if v is a Banach space or not.

OK, so what we're going to do is we're going to use that characterization we had earlier of when a norm space is a Banach space in terms of summability absolute summability. So suppose Tn is a sequence of elements of bounded linear operators such that this constant C, which is the sum of these norms, exists as a real number. So let's suppose that we have an absolutely summable series of linear operators.

And to show that this is a Banach space, we want to show-- we want to show that this series is summable. Then by that theorem we have earlier, that if you have a norm space such that every absolutely summable series is summable, then it's a Banach space. We conclude the proof of the theorem.

And how we're going to show this is summable is, again, kind of the same strategy we used to prove that the space of bounded continuous functions on a metric space is a Banach space. We're going to come up with a candidate for this, show that it's actually a bounded linear operator, and then that the convergence is uniform-- or not uniform, but the convergence is in the space in the operator norm.

OK, so let me just make a-- let me just make a note of something real quick, which I meant to write. You know what? Let's write it up there because that's where it belongs. But I didn't write it down. So let me just make a remark here.

The operator norm is defined for all unit length v. But it automatically gives us a bound. And I'm moving a little quickly because-- maybe quicker than I should have, but-- OK, what was I going to say?

First off, one thing I should have said is that this is an actual finite number if I have a bounded linear operator because, remember, T, a bounded linear operator, implies there exists a constant such that for all v in V, the norm of Tv is less than or equal to a constant times the norm of V. So when v has unit length, this constant bounds-- that satisfies this inequality bounds these numbers for all v with unit length.

And in fact, this supremum is the smallest such C that I can put here. So that's the one comment I wanted to say, that this is an actual, well-defined thing. OK, so I wanted to say that. And then also, now from rescaling, we get a bound for all v in terms of the operator norm.

Then if I take v over its length and take its operator-- or I take its norm-- so this is a unit length vector in V. I applied T to it. So that's always less than or equal to the sup over all norm of T times something, where that something has unit length. But this is a linear operator so that this scalar comes out here, and then comes out of the norm again by homogeneity. And therefore, I get that Tv is that.

So in short, the point of this remark was to say that-- this is not really maybe the best way to say it. But if I think of T acting on v as the product of T and v, then this says the norm of the product of T and v is less than or equal to the product of the norms of T and v.

Sorry if I went a little quick there. Maybe you stopped and were wondering, why is that even a real-- why is that even a real thing? Why is that an actual number? What am I going on about? Maybe I'm just a little overexcited about teaching functional analysis because this is kind of your first adult analysis class, I will say.

OK, so back to the proof at hand, we have this series of absolutely summable bounded linear operators. So the sum of the norms is convergent, meaning this series sum is finite. And we want to show that this series of the T sub n's is summable so that this has a limit in the space of bounded linear operators.

So we're going to come up with a candidate. Let v be in V. If I look at the sum of the norms of T sub n of v-- so T sub n applied to v, this is bounded by-- I mean, I'm writing it like this. But let me be a little more careful.

Let m be a natural number. Then sum from n equals 1 to m of T sub n of v norm, this is less than or equal to the sum from n equals 1 to m of T sub n times the norm of v. And so v, that's just a number. It comes outside the sum. And the sum from n equals 1 to m is bounded by the sum from n equals 1 to infinity of these non-negative numbers. So this is less than or equal to the norm of v times T sub n equals Cv.

So for all m, I've shown that this thing is bounded by C times the norm of v. And therefore, the partial sums corresponding to this series of non-negative real numbers are bounded. And therefore, that series converges. So this sequence of partial sums of non-negative real numbers is bounded, which implies that it converges, which I'll write Tn v converges.

Now, think of T sub n of v as-- so T sub n of v, this is an element in W for each n. So I've shown that the series T sub n of v, this is absolutely summable in W. These are elements of W. And their norms is an absolutely convergent series of real numbers.

Now, since W is a Banach space, every absolutely summable series is summable, therefore summable. We therefore define a map from V to W via T sub v-- or T of v is defined to be the limit as m goes to infinity sum from n equals 1 to m of T sub n applied to V, which we've shown for every v is a convergent series in W. That was the point of everything that came before.

For each v, this is a summable series. And we define its limit, which depends on v, as this map T going from V to W. So this is our candidate, which we'll show is a bounded linear operator. So let's show it's linear.

So T is linear. Why? For all lambda 1, lambda 2 in the space of scalars, RC, v1, v2, in V, we have that T of lambda 1 v1 plus lambda 2 v2-- this is, by definition, equal to limit as m goes to infinity of the sum T sub n lambda 1 v1 plus lambda 2 v2.

And now, each T sub n is a linear operator. So I can write this as limit as m goes to infinity of lambda 1 Tv sub 1 plus lambda 2 Tv sub 2. And now, this thing here converges to T of v as m goes to infinity. This thing here converges to T of v2 as m goes to infinity. And therefore, the limit of the sum is the sum of the limits.

So technically, I did not prove that in a norm space if I have two sequences converging to v1 and v2, then the sum of that sequence converges to v1 plus v2. But it's the exact same proof as in R, all right? Just replace the absolute values with norms. So this should be believable. And therefore, T is a linear operator.

Now, let's prove that it's a bounded linear operator. OK, so now, we'll show that T is a bounded linear operator. Let v be in V norm equal to 1. Then Tv, This is equal to the norm of the limit as m goes to infinity of n equals 1 to m T sub n of v norm.

And norms of limits equal to limit of norms, just like in the case of R, so this is equal to limit as m goes to infinity of-- OK? Now, this is less than or equal to-- by the triangle inequality-- the triangle inequality for two things, this implies a triangle inequality for m things by induction, which is less than or equal to the norm of T sub n times-- I shouldn't even say v. It's just less than or equal to the norm of T sub n.

And this is precisely equal to the sum of the norms, which I called C, this constant C, which we know is finite-- which we assumed is finite, right? So we assumed it was an absolutely summable series of bounded linear operators. So therefore, I've got that Tv is less than or equal to C for all unit length v. And therefore-- which again, by scaling arguments, this was for all v equals 1, which implies by scaling arguments T for all v.

I guess I didn't have to start with norm of v equals 1 if that would have brought a norm of v here. And then that would just be C times the norm of v instead of doing this separate part. So change that in your notes. But I'm up against the clock here, so I'm not going to do this on the board.

OK, so T is an actual bounded linear operator. Now, let's show that the sum of these operators converges to T in the operator norm. So now, we claim that T equals-- oh, that's awful. Tn converges to T as m goes to infinity in the space of bounded linear operators, meaning in the operator norm So I think, for this, this is the reason why I maybe accidentally wrote the norm of v equals 1 there.

So let v be in V with norm of v equals 1. Then T of v minus T sum from n equals 1 to m of T sub n of v in norm. Now, T is equal to the whole sum n equals 1 to infinity. So I'll write this as maybe limit m-prime goes to infinity of sum from n equals 1 to m-prime Tn V minus sum from n equals 1 to m Tn v. And this equals the limit m-prime goes to infinity of sum from n equals m plus 1 m-prime T sub n v norm.

And now, this is less than or equal to the limit as m-prime goes to infinity of-- so the norm-- so this is, in fact, equal to the limit of the norm. And then I use the triangle inequality to bring the norm inside. This is less than or equal to sum equals m plus 1 m-prime T sub n of v. And now, this is less than or equal to limit m-prime goes to infinity of n equals m plus 1 to m-prime T sub n norm because v has unit length. So T sub n applied to v in norm is bounded by the operator norm of T sub n.

And this equals the sum from n equals m plus 1 to infinity of T sub n. Now, what we know-- so this is just a series involving real numbers. And we know if the series converges, then the tails have to go to 0. So this goes to 0 as m goes to infinity, right? Or I shouldn't do that step just yet. Sorry, I'm making mistakes. I'm up against the clock that I see in the back.

So I started off with this quantity here and ended up bounding it by this thing uniformly in V. So that implies that the operator norm of T minus sum from n equals 1 to m of T sub n is less than or equal to the sum from n equals m plus 1 operator norm of T sub n. And this last thing goes to 0 as m goes to infinity because it's the tail of a convergent series of non-negative terms. And therefore, the operator norm of T minus this partial sum goes to 0 as m goes to infinity, and therefore converges to T.

So you see, I mean, it had the same basic format that the last argument did. You found a candidate. You showed it's in the space. And then you showed convergence of the space.

So let me finish with a definition. If V is a norm space, then we denote v-prime as the space of bounded linear operators from V to the space of scalars. This is referred to as the dual space of V. And since the space of scholars is always R or C, both of which are complete-- I mean, they're the simplest examples of Banach spaces-- by the theorem we just proved, since the field of scalars is always complete, the dual space is always a Banach space.

And let me just write here a simple example, which will be in the exercises, that, in a sense, for all p between 1 and strictly less than infinity, I can identify the dual space of little lp as little lp-prime, where p-prime-- this is now just a number bigger than 1, where p-prime and p satisfy this relation.

So in particular, the dual of l1 is l infinity. The dual of l2 is l2. This is very special about l2. But if I take the dual of l infinity, p equals infinity. I would get p-prime equals 1. This, in fact, does not equal little l1. This is something of a headache that manifests itself for the big Lp spaces as well.

And life would be a lot easier if this were the case, that the dual of l infinity was l1. But unfortunately, it's not. And that causes a headache. And l2, little l2, this is the only lp that has this property, that its dual is given by little l2.

All right, and in the exercises, I'll discuss precisely in what way you can identify the duel with this little lp-prime in this way. All right, we'll stop there.