Assignments are to be submitted to Gradescope by 24:00.

1. Let \( f : \mathbb{R} \to \mathbb{R} \). Prove that the collection of sets
\[
\mathcal{A} = \{ E \subseteq \mathbb{R} \mid f^{-1}(E) \text{ is Lebesgue measurable} \}
\]
is a \( \sigma \)-algebra.

2. Let \( E \subseteq \mathbb{R} \), and assume that \( m^*(E) < \infty \). Prove that \( E \) is measurable if and only if for every \( \epsilon > 0 \) there exists a finite union of open intervals \( U \) such that \( m^*(U \Delta E) < \epsilon \). This result is known as Littlewood’s first principle: every measurable set is nearly a finite union of open intervals.

   \textit{Hint:} To prove the converse direction, let \( A \subseteq \mathbb{R} \), and prove that for every \( \epsilon > 0 \),
   \[
   m^*(A \cap E) + m^*(A \cap E^c) \leq m^*(A) + \epsilon.
   \]

   You may use without proof the fact that a finite union of open intervals is measurable. This is covered in Lecture 8 which has been moved to Week 5.

3. Let \( E \) be a measurable set.
   
   (a) Prove that for all \( x \in \mathbb{R} \), \( E + x \) is measurable.
   
   (b) Prove that for all \( r > 0 \), \( rE := \{ ry \mid y \in E \} \) is measurable.
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