Assignments are to be submitted to Gradescope by 24:00.

1. (a) Prove that if E and F are measurable sets then
$$m(E \cup F) + m(E \cap F) = m(E) + m(F).$$
   (b) Prove *continuity from below* for Lebesgue measure: if $\{E_n\}_{n=1}^{\infty}$ is a countable collection of measurable sets, $m(E_1) < \infty$ and
$$E_1 \supset E_2 \supset \cdots ,$$
   then
$$m\left(\bigcap_{k=1}^{\infty} E_k\right) = \lim_{n\to\infty} m(E_n).$$

2. Suppose that E is measurable, and $f : E \to [-\infty, \infty]$ and $g : E \to [-\infty, \infty]$ are measurable.

   (a) Prove that fg is measurable (where $(+\infty)(-\infty) := -\infty, (\pm\infty)(\pm\infty) := +\infty$, and $(\pm\infty) \cdot 0 := 0$).

   (b) Let $a \in \mathbb{R}$. Define $h : E \to [-\infty, \infty]$ via
$$h(x) = \begin{cases} a & \text{if } f(x) = -g(x) = \pm\infty, \\ f(x) + g(x) & \text{otherwise.} \end{cases}$$
   Prove that h is measurable.

3. (a) Suppose that E and F are measurable sets and $f : E \cup F \to [-\infty, \infty]$. Prove that f is measurable if and only if its restrictions $f|_E$ and $f|_F$ are measurable.

   (b) Suppose that E is a measurable set and $f : E \to [-\infty, \infty]$. Let $g : \mathbb{R} \to [-\infty, \infty]$ be the extension of f by zero:
$$g(x) = \begin{cases} f(x) & \text{if } x \in E, \\ 0 & \text{if } x \in E^c. \end{cases}$$
   Prove that f is measurable if and only if g is measurable.

   (c) Suppose that E is a measurable set, and $u : E \to \mathbb{R}$ and $v : E \to \mathbb{R}$ are measurable. Prove that $(u^2 + v^2)^{1/2}$ is measurable.

4. Suppose that $E \subset \mathbb{R}$ is a measurable set such that $m(E) < \infty$, and suppose that $f_n : E \to \mathbb{R}$ is measurable for all n, $f : E \to \mathbb{R}$ is measurable and $\lim_{n\to\infty} f_n(x) = f(x)$ a.e. on E.

(a) For $k, n \in \mathbb{N}$, let

$$F_n(k) = \bigcup_{m=n}^{\infty} \{x \in E \mid |f_m(x) - f(x)| \geq k^{-1}\}$$

Prove that $\lim_{n\to\infty} m(F_n(k)) = 0$.

Hint: Show that $\cap_{n=1}^{\infty} F_n(k)$ is a set of measure zero, and appeal to Problem 1 (b).

(b) Let $\epsilon > 0$. By part (a), for $k \in \mathbb{N}$, there exists $n_k \in \mathbb{N}$ such that $m(F_{n_k}(k)) < 2^{-k}\epsilon$. Let $F = \cup_{k=1}^{\infty} F_{n_k}(k)$. Prove that $m(F) < \epsilon$ and $f_n \to f$ uniformly on F^c. This result is known as *Littlewood's second principle*: every convergent sequence of measurable functions is nearly uniformly convergent.