This exam is open book/open notes including the lecture notes by Richard Melrose, my handwritten lecture notes, the typed notes by Andrew Lin, Real Analysis by Royden (if you bought a copy), solutions to the assignments, Piazza threads and recorded lectures.

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1. (8 points) Let \( a < b \). We say \( f \in C([a,b]) \) is Lipschitz continuous if there exists \( L \geq 0 \) such that for all \( x, y \in [a,b] \)

\[
|f(x) - f(y)| \leq L|x - y|.
\]

Let \( \Lambda([a,b]) = \{ f \in C([a,b]) \mid f \text{ is Lipschitz continuous} \} \) be the normed space of Lipschitz continuous functions with norm

\[
\|f\| := \sup_{x \in [a,b]} |f(x)| + \sup_{x \neq y \in [a,b]} \frac{|f(x) - f(y)|}{|x - y|}.
\]

Prove that \( \Lambda([a,b]) \) is a Banach space. You may use without proof the fact proved in class that \( C([a,b]) \) with norm \( \|f\|_{\infty} := \sup_{x \in [a,b]} |f(x)| \) is a Banach space.

2. (8 points) Recall that \( c_0 = \{b = \{b_n\}_n \in \ell^\infty \mid \lim_{n \to \infty} b_n = 0\} \) is a closed subspace of \( \ell^\infty \). Prove that for all \( a = \{a_n\}_n \in \ell^\infty \)

\[
\|a + c_0\|_{\ell^\infty / c_0} := \inf_{b \in c_0} \|a + b\|_\infty = \limsup_{n \to \infty} |a_n|.
\]

Here we recall that

\[
\limsup_{n \to \infty} |a_n| := \inf_{n \geq 1} \left[ \sup_{k \geq n} |a_k| \right].
\]

**Hint:** First use the definitions to show that \( \|a + c_0\|_{\ell^\infty / c_0} \leq \limsup_{n} |a_n| \). To show the reverse inequality, prove that for all \( \epsilon > 0 \), \( \limsup_{n} |a_n| < \|a + c_0\|_{\ell^\infty / c_0} + \epsilon \). You may use without proof the fact from 18.100 that if \( a \in \ell^\infty \) and \( b \in c_0 \) then

\[
\limsup_{n \to \infty} |a_n + b_n| = \limsup_{n \to \infty} |a_n|.
\]

3. (a) (4 points) Suppose that \( V \) and \( W \) are Banach spaces. Prove that if \( \{T_n\}_n \) is a sequence in \( \mathcal{B}(V,W) \) and \( T : V \to W \) is a linear map such that for all \( x \in V \),

\[
\lim_{n \to \infty} T_n x = Tx,
\]

then \( T \) is a bounded linear operator.

**Remark:** Note that we are not assuming that \( \{T_n\}_n \) is converging to the linear map \( T \) in operator norm.
(b) (4 points) Let $\cdot_1$ and $\cdot_2$ be two norms on a vector space $V$ such that for all $v \in V$, $\|v\|_1 \leq \|v\|_2$. Prove that if $V$ is a Banach space with respect to both norms, then there exists $C > 0$ such that for all $v \in V$, $\|v\|_2 \leq C\|v\|_1$.

*Hint:* Consider the identity as a mapping between two Banach spaces, $I : (V, \cdot_1) \to (V, \cdot_2)$.

4. Let $E \subset \mathbb{R}$ be measurable. For a measurable function $f : E \to \mathbb{R}$, define the *essential supremum* of $f$,

$$\|f\|_\infty := \inf \{C \geq 0 \mid m(\{x \in E \mid |f(x)| > C\}) = 0\}.$$ 

(a) (4 points) Prove that if $f : E \to \mathbb{R}$ is a measurable function then

$$|f(x)| \leq \|f\|_\infty$$

almost everywhere on $E$.

*Hint:* For all $n \in \mathbb{N}$, there exists a measurable set $F_n \subset E$ with $m(F_n) = 0$ such that for all $x \in E \setminus F_n$,

$$|f(x)| \leq \|f\|_\infty + \frac{1}{n}.$$ 

Now consider $x \in E \setminus (\bigcup_n F_n)$.

(b) (4 points) Prove that if $f : E \to \mathbb{R}$ and $g : E \to \mathbb{R}$ are measurable functions and $c \in \mathbb{R}$ then

$$\|cf\|_\infty = |c|\|f\|_\infty,$$

$$\|f + g\|_\infty \leq \|f\|_\infty + \|g\|_\infty.$$ 

5. Let $\varphi \in L^+(E)$ be a simple function with standard representation

$$\varphi = \sum_{i=1}^{n} a_i \chi_{A_i},$$

where for each $i$, $A_i \subset E$ is measurable, if $i \neq j$ then $A_i \cap A_j = \emptyset$, and $\cup_{i=1}^{n} A_i = E$.

(a) (4 points) Prove that if $F \subset E$ is measurable then

$$\int_F \varphi = \int_E \varphi \chi_F.$$ 

*Hint:* Verify and use the simple fact that $\chi_{A_i \cap F} = \chi_{A_i} \chi_F$.

(b) (4 points) Let $F, G \subset E$ be measurable with $F \cap G = \emptyset$. Prove that

$$\int_{F \cup G} \varphi = \int_F \varphi + \int_G \varphi.$$
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