This exam is open book/open notes including the lecture notes by Richard Melrose, my handwritten lecture notes, the typed notes by Andrew Lin, *Real Analysis* by Royden (if you bought a copy), solutions to the assignments, Piazza threads and recorded lectures.

However, **collaborating with other students or the internet is strictly prohibited**. Evidence to the contrary will be treated as academic misconduct and will be responded to according to MIT Institute Policy 10.2.

1. (8 points) Let $a < b$. We say $f \in C([a,b])$ is *Lipschitz continuous* if there exists $L \geq 0$ such that for all $x, y \in [a,b]$

$$|f(x) - f(y)| \leq L|x - y|.$$

Let $\Lambda([a,b]) = \{f \in C([a,b]) \mid f \text{ is Lipschitz continuous}\}$ be the normed space of Lipschitz continuous functions with norm

$$\|f\| := \sup_{x\in[a,b]} |f(x)| + \sup_{x\neq y\in[a,b]} \frac{|f(x)-f(y)|}{|x-y|}.$$

Prove that $\Lambda([a,b])$ is a Banach space. You may use without proof the fact proved in class that $C([a,b])$ with norm $\|f\|_\infty := \sup_{x\in[a,b]} |f(x)|$ is a Banach space.

2. (8 points) Recall that $c_0 = \{b = \{b_n\}_n \in \ell^\infty \mid \lim_{n\to\infty} b_n = 0\}$ is a closed subspace of ℓ^∞. Prove that for all $a = \{a_n\} \in \ell^\infty$

$$\|a + c_0\|_{\ell^\infty/c_0} := \inf_{b\in c_0} \|a + b\|_\infty = \limsup_{n\to\infty} |a_n|.$$

Here we recall that

$$\limsup_{n\to\infty} |a_n| := \inf_{n\geq 1} \left[\sup_{k\geq n} |a_k|.\right]$$

Hint: First use the definitions to show that $\|a + c_0\|_{\ell^\infty/c_0} \leq \limsup_n |a_n|$. To show the reverse inequality, prove that for all $\epsilon > 0$, $\limsup_n |a_n| < \|a + c_0\|_{\ell^\infty/c_0} + \epsilon$. You may use without proof the fact from 18.100 that if $a \in \ell^\infty$ and $b \in c_0$ then

$$\limsup_{n\to\infty} |a_n + b_n| = \limsup_{n\to\infty} |a_n|.$$

3. (a) (4 points) Suppose that V and W are Banach spaces. Prove that if $\{T_n\}_n$ is a sequence in $\mathcal{B}(V,W)$ and $T : V \to W$ is a linear map such that for all $x \in V$,

$$\lim_{n\to\infty} T_n x = Tx,$$

then T is a bounded linear operator.

Remark: Note that we are **not** assuming that $\{T_n\}_n$ is converging to the linear map T in operator norm.

(b) (4 points) Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be two norms on a vector space V such that for all $v \in V$, $\|v\|_1 \leq \|v\|_2$. Prove that if V is a Banach space with respect to both norms, then there exists $C > 0$ such that for all $v \in V$, $\|v\|_2 \leq C\|v\|_1$.

Hint: Consider the identity as a mapping between two Banach spaces, $I : (V, \|\cdot\|_2) \to (V, \|\cdot\|_1)$.

4. Let $E \subset \mathbb{R}$ be measurable. For a measurable function $f : E \to \mathbb{R}$, define the *essential supremum* of f,

$$\|f\|_\infty := \inf \{C \geq 0 \mid m(\{x \in E \mid |f(x)| > C\}) = 0\}.$$

(a) (4 points) Prove that if $f : E \to \mathbb{R}$ is a measurable function then

$$|f(x)| \leq \|f\|_\infty$$

almost everywhere on E.

Hint: For all $n \in \mathbb{N}$, there exists a measurable set $F_n \subset E$ with $m(F_n) = 0$ such that for all $x \in E \backslash F_n$,

$$|f(x)| \leq \|f\|_\infty + \frac{1}{n}.$$

Now consider $x \in E \backslash (\cup_n F_n)$.

(b) (4 points) Prove that if $f : E \to \mathbb{R}$ and $g : E \to \mathbb{R}$ are measurable functions and $c \in \mathbb{R}$ then

$$\|cf\|_\infty = |c|\|f\|_\infty,$$
$$\|f + g\|_\infty \leq \|f\|_\infty + \|g\|_\infty.$$

5. Let $\varphi \in L^+(E)$ be a simple function with standard representation

$$\varphi = \sum_{i=1}^{n} a_i \chi_{A_i},$$

where for each i, $A_i \subset E$ is measurable, if $i \neq j$ then $A_i \cap A_j = \varnothing$, and $\cup_{i=1}^n A_i = E$.

(a) (4 points) Prove that if $F \subset E$ is measurable then

$$\int_F \varphi = \int_E \varphi \chi_F.$$

Hint: Verify and use the simple fact that $\chi_{A_i \cap F} = \chi_{A_i} \chi_F$.

(b) (4 points) Let $F, G \subset E$ be measurable with $F \cap G = \varnothing$. Prove that

$$\int_{F \cup G} \varphi = \int_F \varphi + \int_G \varphi.$$