1. Let $H$ be a Hilbert space, and let $\{e_n\}_{n=1}^{\infty}$ be a countably infinite orthonormal subset of $H$. Let $W = \text{span} \{e_n\}_{n=1}^{\infty}$ be the subspace of all \textbf{finite} linear combinations of these vectors.

(a) Prove that $w \in \overline{W}$ (the closure of $W$) if and only if there exists $\{c_n\}_{n=1}^{\infty} \in \ell^2$ such that

$$w = \sum_{n=1}^{\infty} c_n e_n.$$ 

(b) Let $u \in H$. Prove that for all $w \in \overline{W}$,

$$u - \sum_{n=1}^{\infty} \langle u, e_n \rangle e_n \leq \|u - w\|$$

with equality if and only if $w = \sum_{n=1}^{\infty} \langle u, e_n \rangle e_n$.

\textit{Hint}: If $w = \sum_{n=1}^{\infty} c_n e_n$, then compute $\|u - w\|^2$ and complete the square.

2. Let $H$ be a Hilbert space, and let $W$ be a linear subspace of $H$.

(a) Prove that

$$W^\perp := \{u \in H \mid \langle u, w \rangle = 0 \text{ for all } w \in W\}$$

is a closed linear subspace of $H$.

(b) Prove that $\overline{W} = (W^\perp)^\perp$.

3. Let $s \geq 0$. We say $f \in L^2([-\pi, \pi])$ is an element of the \textit{Sobolev space} $H^s(\mathbb{T})$ of order $s$ if

$$\lim_{N \to \infty} \sum_{|n| \leq N} |\hat{f}(n)|^2 (1 + |n|^2)^s < \infty.$$ 

(a) Suppose that $f \in C^k([-\pi, \pi])$ and

$$f(\pi) = f(-\pi), f'(\pi) = f'(-\pi), \ldots, f^{(k-1)}(\pi) = f^{(k-1)}(-\pi).$$

Prove that $f \in H^s(\mathbb{T})$ for all $0 \leq s \leq k$.

\textit{Hint}: Integrate by . . .

(b) Prove that $H^s(\mathbb{T})$ is a vector space and

$$\langle f, g \rangle_{H^s(\mathbb{T})} := \lim_{N \to \infty} \sum_{|n| \leq N} \hat{f}(n) \overline{\hat{g}(n)} (1 + |n|^2)^s, \quad f, g \in H^s,$$

is a Hermitian inner product on $H^s(\mathbb{T})$.

(c) Prove that $H^s(\mathbb{T})$ is a Hilbert space.
4. (Sobolev embedding) Let $s > 1/2$. The purpose of this exercise is to show that $H^s(\mathbb{T})$ embeds into $C([-\pi, \pi])$ continuously.

(a) Prove that there exists a constant $C(s) > 0$ such that for all $f \in H^s(\mathbb{T})$

$$\lim_{N \to \infty} \sum_{|n| \leq N} |\hat{f}(n)| \leq C(s)\|f\|_{H^s(\mathbb{T})}.$$ 

(b) Prove that if $f \in H^s(\mathbb{T})$ then $f \in C([-\pi, \pi])$ (strictly speaking, $\exists g \in C([-\pi, \pi])$ such that $f = g$ a.e.), and moreover,

$$\|f\|_\infty \leq C(s)\|f\|_{H^s(\mathbb{T})}.$$ 

Hint: It may be useful to recall the Weierstrass M-test from 18.100.