This exam is open book/open notes including the lecture notes by Richard Melrose, my handwritten lecture notes, the typed notes by Andrew Lin, *Real Analysis* by Royden (if you bought a copy), solutions to the assignments, Piazza threads and recorded lectures.

However, **collaborating with other students or the internet is strictly prohibited**. Evidence to the contrary will be treated as academic misconduct and will be responded to according to MIT Institute Policy 10.2.

1. (8 points) Determine the Fourier coefficients

$$\hat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx, \quad n \in \mathbb{Z},$$

where $e^{it} = \cos t + i \sin t$, for the function

$$f(x) = \begin{cases} -1 & \text{if } x \in [-\pi, 0], \\ 1 & \text{if } x \in (0, \pi], \end{cases}$$

and use the result to compute $\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2}$.

Hint: What is the relationship between $\|f\|_2^2$ and $\sum_n |\hat{f}(n)|^2$?

2. (8 points) Use the appropriate convergence theorems to compute

$$\lim_{n\to\infty}\int_0^1 \frac{n\cos x}{1+n^2x^2}dx,$$
$$\lim_{n\to\infty}\int_0^\infty n\sin(x/n)[x(1+x^2)]^{-1}dx.$$

Hint: A change of variables may be useful in one or both of the integrals before computing the limit.

3. (8 points) Let H be a Hilbert space, and let $W \subset H$ be a linear subspace. Prove that $W^\perp = \{u \in H \mid \langle u, w\rangle = 0 \ \forall w \in W\}$ is a closed linear subspace of H and $(W^\perp)^\perp = \overline{W}$.

4. Let $\{\mu_k\}_k$ be a bounded sequence of complex numbers. Define

$$Ma = \{\mu_k a_k\}_k, \quad a = \{a_k\}_k \in \ell^2.$$

Then $M \in \mathcal{B}(\ell^2)$ and $\|M\| \le \sup_k |\mu_k|$.

(a) (4 points) Prove that M is a self-adjoint operator on ℓ^2 if and only if $\mu_k \in \mathbb{R}$ for all $k \in \mathbb{N}$.

(b) (4 points) Assume now that $\lim_{k\to\infty}\mu_k = 0$. Prove that M is a compact operator on ℓ^2.

If need be (depending on your approach), you may use without proof the fact (discussed in Piazza) that if $A \in \mathcal{B}(\ell^2)$, then $\overline{\{Ab \mid \|b\|_2 \le 1\}} = \{Ab \mid \|b\|_2 \le 1\}$.

5. (8 points) For $f \in L^2([0,1])$, define

$$Mf(x) = xf(x), \quad x \in [0,1].$$

Then $M \in \mathcal{B}(L^2([0,1]))$ and $\|M\| \leq \sup_{x\in[0,1]} |x| = 1$. Prove that $\mathrm{Spec}(M) = [0,1]$, and M has no eigenvalues.

Hint: To prove that $\lambda \in [0,1] \implies \lambda \in \mathrm{Spec}(M)$, consider: is $1 \in \mathrm{Range}(M - \lambda I)$?

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