April 15, 2021

We'll continue the discussion of Fourier series today – last time, we defined the Fourier coefficients

$$\hat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-int}dt$$

for any function $f \in L^2([-\pi,\pi])$, which we can think of as the L^2 inner product of f with e^{-int} up to a constant. Defining the Nth partial sums

$$S_N f(x) = \sum_{n=-N}^{N} \hat{f}(n)e^{inx},$$

we wanted to know whether $S_N f$ always converges to f in L^2 – that is, whether for all $f \in L^2([-\pi,\pi])$ we have $\lim_{N\to\infty} ||f - S_N f||_2 = 0$.

Based on our discussion of Hilbert spaces, this question is equivalent to asking whether a function $f \in L^2([-\pi,\pi])$ with all Fourier coefficients zero must be the zero function (since we're trying to ask whether $\left\{\frac{1}{\sqrt{2\pi}}e^{inx}\right\}_{n\in\mathbb{Z}}$ is a maximal orthonormal subset). Our main step last time was to define the Cesaro-Fourier mean

$$\sigma_N f(x) = \frac{1}{N+1}\sum_{k=0}^{N} S_k f(x),$$

hoping that means of sequences converge better than the sequences themselves. Our goal is then to show that $||\sigma_N f - f||_2 \to 0$ as $N \to \infty$, and that will give us the desired convergence result for Fourier series.

We'll first rewrite the partial Fourier sums slightly differently, much like how we previously used the Dirichlet kernel:

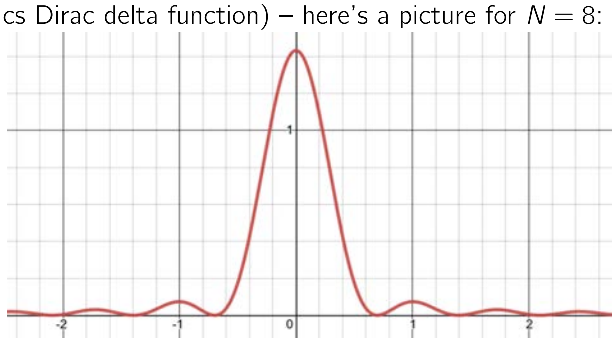
Proposition 173

For all $f \in L^2([-\pi,\pi])$, we have

$$\sigma_N f(x) = \int_{-\pi}^{\pi} K_N(x-t)f(t)dt, \quad K_N(x) = \begin{cases} \frac{N+1}{2\pi} & x = 0 \\ \frac{1}{2\pi(N+1)}\left(\frac{\sin\left(\frac{N+1}{2}x\right)}{\sin\frac{x}{2}}\right)^2 & \text{otherwise.} \end{cases}$$

The function $K_N(x)$ is called the **Fejér kernel**, and it has the following properties: **(1)** $K_N(x) \geq 0$ and $K_N(x) = K_N(-x)$ for all x, **(2)** K_N is periodic with period 2π, **(3)** $\int_{-\infty}^{\infty} K_N(t)dt = 1$, and **(4)** for any $\delta \in (0,\pi)$ and for all $\delta \leq |x| \leq \pi$, we have $|K_N(x)| \leq \frac{1}{2\pi(N+1)\sin^2\frac{\delta}{2}}$.

The idea is that the Fejér kernel grows more and more concentrated at the origin as $N \to \infty$, but the area of the curve is always 1 (like the physics Dirac delta function) – here's a picture for $N = 8$:



The reason we might believe that these Cesaro means converge to f is that

$$\sigma_N f(x) = \int_{-\pi}^{\pi} K_N(x-t) f(t) dt,$$

and K_N is very sharply peaked around $t = x$, so as N gets larger and larger, the main contribution to the integral comes from $f(x) \approx f(t)$ if f is well-behaved enough. So then we end up with

$$\approx f(x) \int_{-\pi}^{\pi} K_N(x-t) dt = f(x) \cdot 1,$$

since k_N evaluates to the same over any interval of length 2π by periodicity. So that's a heuristic motivation for working with the Cesaro means here! (Some of these properties also applied when we did a similar procedure with our partial sums $S_N f(x)$, but the **Dirichlet kernel is not nonnegative** – that difference actually makes a big difference in the final proof.)

Proof. Recall that

$$S_k f(x) = \int_{-\pi}^{\pi} D_k(x-t) f(t) dt$$

for the Dirichlet kernel

$$D_k(t) = \begin{cases} \frac{2N+1}{2\pi} & t = 0 \\ \frac{1}{2\pi} \frac{\sin\left(\left(N+\frac{1}{2}\right)t\right)}{\sin\frac{t}{2}} & \text{otherwise.} \end{cases}$$

We can use this fact to find that

$$\sigma_N f(x) = \frac{1}{N+1} \sum_{k=0}^{N} S_k f(x) = \int_{-\pi}^{\pi} \frac{1}{N+1} \sum_{k=0}^{N} D_k(x-t) f(t) dt,$$

and thus we know that the desired kernel is

$$K_N(x-t) = \frac{1}{N+1} \sum_{k=0}^{N} D_k(x-t).$$

We can now substitute in our expression for D_k, using the variable x instead of $x - t$. The case $x = 0$ can be done easily (we just have constants), and for all other x we can slightly rewrite our expression as

$$K_N(x) = \frac{1}{2\pi(N+1)} \frac{1}{2\left(\sin\frac{x}{2}\right)^2} \sum_{k=0}^{N} 2 \sin\frac{x}{2} \sin\left(\left(k + \frac{1}{2}\right) x\right).$$

By the trig product-to-sum identity, this simplifies to

$$= \frac{1}{2\pi(N+1)} \frac{1}{2\left(\sin\frac{x}{2}\right)^2} \sum_{k=0}^{N} \cos(kx) - \cos\left((k+1)x\right),$$

and this is a telescoping sum which simplifies to

$$= \frac{1}{2\pi(N+1)} \frac{1}{2\left(\sin\frac{x}{2}\right)^2} \left(1 - \cos((N+1)x)\right).$$

We can now use another trig formula $\frac{1-\cos x}{2} = \cos^2\left(\frac{x}{2}\right)$ to get

$$= \frac{1}{2\pi(N+1)} \frac{1}{\left(\sin\frac{x}{2}\right)^2} \sin^2\left(\frac{N+1}{2} x\right),$$

which is indeed the expression for our Fejér kernel.

We can now verify the properties of the Fejér kernel directly: **(1)** is true because we have a manifestly positive expression and $\sin^2(cx)$ is even, and **(2)** is true because $\sin^2$ is also periodic with half the period of the corresponding sin. For **(3)**, notice that

$$\int_{-\pi}^{\pi} D_k(t)dt = \int_{-\pi}^{\pi} \sum_{n=-k}^{k} e^{int}dt,$$

and the integral of e^{int} is zero unless $n = 0$ (by 2π-periodicity), so we just pick up the $n = 0$ term and get 1. Since σ_N is the average of the D_ks, the integral of σ_N is also the average of the average of the D_ks, which will also be 1.

Finally, for **(4)**, notice that $\sin^2 \frac{x}{2}$ is an even function which is increasing on $[0, \pi]$. So if we pick some $\delta \in (0, \pi)$, we can say that

$$\delta \le |x| \le \pi \implies \sin^2 \frac{x}{2} \ge \sin^2 \frac{\delta}{2},$$

so we indeed get the expected

$$K_N(x) = |K_N(x)| \le \frac{1}{2\pi(N+1)\sin^2 \frac{\delta}{2}} \sin^2\left(\frac{N+1}{2}x\right) \le \frac{1}{2\pi(N+1)\sin^2 \frac{\delta}{2}}.$$

□

Now, we can prove convergence of the Cesaro means $\sigma_N f$ to f by first doing it for continuous functions – we showed that the continuous functions with endpoints 0 are dense in L^2 (so we can show convergence appropriately), and continuous functions with endpoints both 0 can indeed be treated as 2π-periodic. So the subspace of 2π-periodic continuous functions is dense in L^2, and we'll consider this dense subset first because it's where the heuristic argument we made above applies rigorously.

Theorem 174 (Fejér)
Let $f \in C([-\pi, \pi])$ be 2π-periodic (so $f(-\pi) = f(\pi)$). Then $\sigma_N f \to f$ uniformly on $[-\pi, \pi]$.

In other words, we have an even stronger result than L^2 convergence, now that we're limiting ourselves to continuous functions and have the stronger uniform norm. But this does **not** imply that the Fourier series of f converges pointwise to f – there are indeed Fourier series representations of continuous functions that diverge at a point. Instead, it's the Cesaro mean and the Fejér kernel that help us out here!

Proof. First, we extend f to all of $\mathbb{R}$ by periodicity (defining it so that $f(x + 2\pi) = f(x)$ for all $x \in \mathbb{R}$). Our function is then an element of $C(\mathbb{R})$ (still continuous), and it is 2π-periodic, so it is uniformly continuous and bounded on all of $\mathbb{R}$ (that is, $||f||_\infty = \sup_{x\in[-\pi,\pi]} f(x) < \infty$).

We wish to show that $\sigma_N f$ converge uniformly on f, which means that for all $\varepsilon > 0$ we need to find an M so that for all $n \ge M$, we have $|\sigma_N f(x) - f(x)| < \varepsilon$ for all x. Indeed, for any $\varepsilon > 0$, by uniform continuity of f, there exists some $\delta > 0$ so that for all $|y - z| < \delta$, we have $|f(y) - f(z)| < \frac{\varepsilon}{2}$. So now we can choose $M \in \mathbb{N}$ so that for all $N \ge M$, we have

$$\frac{2||f||_\infty}{(N+1)\sin^2 \frac{\delta}{2}} < \frac{\varepsilon}{2}.$$

(we can do this because the left-hand side converges to 0 as $N \to \infty$). Now because f and K_N are 2π-periodic, we can write the Cesaro mean as

$$\sigma_N f(x) = \int_{-\pi}^{\pi} K_N(x-t)f(t)dt = \int_{x-\pi}^{x+\pi} K_N(\tau)f(x-\tau)d\tau$$

by a change of variables (which is allowed because we're doing integrals over continuous functions, and thus we can use the Riemann integral), and now we have the product of 2π-periodic functions, so the integral of that is the same over any interval of length 2π: switching back to t from τ,

$$= \int_{-\pi}^{\pi} K_N(t) f(x-t) dt.$$

We can now say that for all $N \geq M$ and for all $x \in [\pi, \pi]$, we have

$$|\sigma_N f(x) - f(x)| = \left| \int_{-\pi}^{\pi} K_N(t) f(x-t) dt - \int_{-\pi}^{\pi} K_N(t) f(x) dt \right|$$

where we've added in a $\int_{-\pi}^{\pi} K_N(t)dt$ integral to the $f(x)$ term, which is okay because $f(x)$ doesn't talk to the t-integral. Combining the integrals by linearity gives us

$$= \left| \int_{-\pi}^{\pi} K_N(t) \left(f(x-t) - f(x) \right) dt \right|.$$

We'll use the triangle inequality and then split this integral into two parts now:

$$\leq \int_{-\pi}^{\pi} |K_N(t) \left(f(x-t) - f(x) \right)| dt = \int_{|t| \leq \delta} K_N(t) \left| f(x-t) - f(x) \right| dt + \int_{\delta \leq |t| \leq \pi} K_N(t) \left| f(x-t) - f(x) \right| dt$$

(also using the fact that K_N is always nonnegative). And now we can use our bounds above to simplify this: for the first term, we know that $|(x-t) - x| < \delta$ over the bounds of integration, so $|f(x-t) - f(x)| < \frac{\varepsilon}{2}$. And for the second term, we know that $|f(x-t) - f(x)| < 2||f||_\infty$ because both $f(x-t)$ and $f(x)$ have magnitude at most $||f||_\infty$ for a continuous function, and when $|t| > \delta$ we can use condition **(4)** of the Fejér kernel. Putting this all together, we find the inequality

$$< \frac{\varepsilon}{2} \int_{|t| \leq \delta} K_N(t) dt + \frac{2||f||_\infty}{2\pi(N+1)\sin^2 \frac{\delta}{2}} \int_{\delta \leq |t| \leq \pi} K_N(t) dt.$$

We can now bound both integrals here by the integral over the entire region to get

$$\leq \frac{\varepsilon}{2} + \frac{2||f||_\infty}{(N+1)\sin^2 \frac{\delta}{2}} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

by our choice of N. So we've indeed shown uniform convergence – $\sigma_N f$ is eventually close enough to f for large enough N – and we're done. □

Remark 175. *This same proof can be modified if instead of knowing that $K_n(x) \geq 0$ (which we know for the Fejér kernel), we have that*

$$\sup_N \int_{-\pi}^{\pi} |K_N(x)| < \infty.$$

Then we can show the same uniform convergence by modifying our proof above. But if we try to plug in our Dirichlet kernel here, the condition is not satisfied, since

$$\int_{-\pi}^{\pi} |D_N(x)| dx \sim \log N.$$

So having "almost all of the properties" isn't enough for us to get the analogous results for the Dirichlet kernel!

Now that we've proven that the Cesaro means of a continuous function converge uniformly to that function, we want to show that the Cesaro means of an L^2 function converge to an L^2 function, which would show the condition on the Hilbert space that we want and show convergence of the Fourier series as well. We'll first need the following result:

Proposition 176

For all $f \in L^2([-\pi, \pi])$, we have $||\sigma_N f||_2 \le ||f||_2$.

Proof. We'll first do this for 2π-periodic functions. First suppose that $f \in C([-\pi, \pi])$ is 2π-periodic – extend f to all of $\mathbb{R}$ as before, and then the Cesaro mean is $\sigma_N f(x) = \int_{-\pi}^{\pi} f(x-t)K_N(t)dt$. Thus, we can write out

$$\boxed{||\sigma_N f||_2^2} = \int_{-\pi}^{\pi} |\sigma_N f(x)|^2 dx = \int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(x-s)\overline{f(x-t)}K_N(s)K_N(t)dsdtdx.$$

All of these functions are continuous, so we can change the order of integration by Fubini's theorem to get

$$= \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} K_N(s)K_N(t)\left[\int_{-\pi}^{\pi} f(x-s)\overline{f(x-t)}dx\right] dsdt.$$

By Cauchy-Schwarz, this can be bounded by

$$\le \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} K_N(s)K_N(t)||f(\cdot - s)||_2||f(\cdot - t)||_2 dsdt,$$

where $f(\cdot - s)$ denotes the function that maps $x \mapsto f(x-s)$. And now we're integrating a periodic function $f(\cdot - s)$ over an interval of length 2π, so we can replace that expression with $||f||_2$ (just shifting to another length 2π interval). Doing the same with $f(\cdot - t)$ gives us

$$= ||f||_2^2 \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} K_N(s)K_N(t)dsdt = \boxed{||f||_2^2},$$

because the integral of K_N is 1. This gives us the desired inequality for 2π-periodic functions, and now to extend it to all functions in L^2, suppose we have some general $f \in L^2$. From exercises, we know that there exists a sequence $\{f_n\}_n$ of 2π-periodic continuous functions that converge to f in L^2, meaning that $||f_n - f||_2 \to 0$. So from the definition of the Cesaro means, this means that $||\sigma_N f_n - \sigma_N f||_2 \to 0$ for any fixed N and as $N \to \infty$, leading us to

$$||\sigma_N f||_2 = \lim_{n\to\infty} ||\sigma_N f_n||_2 \le \lim_{n\to\infty} ||f_n||_2$$

(using the 2π-periodic case), and this last result is $||f||_2$ because f_n converges to f in L^2. ☐

So now we're almost done, and combining the two results above will give us what we want:

Theorem 177

For all $f \in L^2$, $||\sigma_N f - f||_2 \to 0$ as $N \to \infty$. Therefore, if $\hat{f}(n) = 0$ for all n, then $f = 0$ (since $\sigma_N f = 0$ for all N).

Proof. (We only need to prove the result in the first sentence – the second follows directly as stated.) Let $f \in L^2([-\pi, \pi])$, and let $\varepsilon > 0$. By density of the 2π-periodic continuous functions, there exists some 2π-periodic $g \in C([-\pi, \pi])$ so that $||f - g||_2 < \frac{\varepsilon}{3}$. Because $\sigma_N g \to g$ uniformly on $[-\pi, \pi]$, there exists some M so that for all $N \ge M$ and for all $x \in [-\pi, \pi]$, we have $|\sigma_N g(x) - g(x)| < \frac{\varepsilon}{3\sqrt{2\pi}}$.

Now for all $N \ge M$, the triangle inequality tells us that

$$||\sigma_N f - f||_2 \le ||\sigma_N f - \sigma_N g||_2 + ||\sigma_N g - g||_2 + ||g - f||_2.$$

The first term is $||\sigma_N(f - g)||_2$ (we can check this from the definition), and by Proposition 176, that is less than $||f - g||_2 < \frac{\varepsilon}{3}$. Meanwhile, the last term is also bounded by $\frac{\varepsilon}{3}$, and the middle term is $\left(\int_{-\pi}^{\pi} |\sigma_N g(x) - g(x)|^2 dx\right)^{1/2} <$

$\left(2\pi \cdot \left(\frac{\varepsilon}{3\sqrt{2\pi}}\right)^2\right)^{1/2} = \frac{\varepsilon}{3}$. So putting this all back into our expression gives us

$$||\sigma_N f - f||_2 < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon,$$

completing the proof. □

So we've now seen a concrete application of the general machinery we've built up for Hilbert spaces! In summary, we've shown that the normalized exponentials form a maximal orthonormal set, so that the partial Fourier sums of f converge to f in L^2. But as previous mentioned, we don't have pointwise convergence everywhere – instead, we can only say that there is a **subsequence** that converges to f pointwise. And in fact, **Carleson's theorem** is a deep result in analysis that tells us that for all $f \in L^2$, $S_N f(x) \to f(x)$ **almost everywhere**.

We can also ask questions about the convergence of Fourier series in other L^p spaces, since all of the definitions also make sense there. It is known additionally that for all $1 < p < \infty$, we always have $||S_N f - f||_p \to 0$, and that this is false for $p = 1, \infty$. But deeper harmonic analysis is needed to prove statements like this, and in particular we would need to learn how to work with **singular integral operators**.

In this class, though, this is as far as we'll go with Fourier series, and next time, we'll move on to the topic of **minimizers over closed convex sets** and (as a consequence) how to identify the dual of a Hilbert space with the Hilbert space itself in a canonical way.