18.103 Final Review 2013

The Fourier transform on $\mathbb{R}$ is defined for all $f \in L^1(\mathbb{R})$ by $\hat{f}(t) = \int_{\mathbb{R}} f(x)e^{-itx}dx$. Denoting $G(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}$, we have

$$\hat{G}(t) = e^{-t^2/2}$$

Main Approximate Identity Lemma. Let $K \in L^1(\mathbb{R})$ satisfy

$$\int_{\mathbb{R}} K(x)\,dx = 1$$

and denote $K_a(x) = (1/a)K(x/a)$, $a > 0$. Then for every $x \in \mathbb{R}$ and $f \in C_0(\mathbb{R})$,

$$\lim_{a\to 0^+} f * K_a(x) = f(x)$$

Theorem 1 (Fourier inversion on $\mathcal{S}$). The mappings T_1 and T_2 defined for f and g in $\mathcal{S}(\mathbb{R})$ by the Riemann integrals

$$(T_1 f)(t) = \int_{\mathbb{R}} f(x)e^{-itx}dx; \quad (T_2 g)(x) = \frac{1}{2\pi}\int_{\mathbb{R}} g(t)e^{itx}dt$$

send the Schwartz class $\mathcal{S}$ to itself. Moreover, the compositions T_2T_1 and T_1T_2 are both the identity mapping on $\mathcal{S}$.

Theorem 2 (Plancherel).

a) For all f and g in $\mathcal{S}$,

$$\|T_1 f\|^2 = 2\pi\|f\|^2; \quad 2\pi\|T_2 g\|^2 = \|g\|^2$$

where

$$\|f\|^2 = \int_{\mathbb{R}} |f(x)|^2 dx$$

b) T_1 and T_2 have unique extensions from $\mathcal{S}$ to continuous mappings from $L^2(\mathbb{R})$ to itself, T_1T_2 and T_2T_1 are the identity mapping on $L^2(\mathbb{R})$ and the properties of part (a) are valid for all f and g in $L^2(\mathbb{R})$.

Theorem 3 (Fourier inversion with truncation). Let $f \in L^2(\mathbb{R})$, and denote

$$s_N(x) = \frac{1}{2\pi}\int_{-N}^{N} \hat{f}(\xi)e^{ixt}\,dt$$

Then

$$\lim_{N\to\infty}\int_{\mathbb{R}} |s_N(x) - f(x)|^2 dx = 0$$

Proposition. If $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, then

$$T_1 f(t) = \int_{\mathbb{R}} f(x)e^{-ixt}\,dx \quad T_2 g(x) = \frac{1}{2\pi}\int_{\mathbb{R}} g(t)e^{ixt}\,dt$$

Theorem 4. Let

$$f * g(x) = \int_{\mathbb{R}} f(x-y)g(y)dy$$

If $f \in L^1(\mathbb{R})$ and $g \in L^1(\mathbb{R})$ or (and this requires more work) if $f \in L^2(\mathbb{R})$ and $g \in L^2(\mathbb{R})$, then

$$\widehat{(f * g)}(t) = \hat{f}(t)\hat{g}(t)$$

Review Problems

1.

a) Find the Fourier series of the function

$$f(x) = \begin{cases} 1, & 0 < x < \pi; \\ 0, & -\pi < x < 0. \end{cases}$$

extended periodically with period 2π. Pay attention to three cases $n = 0$ and $n \neq 0$ odd and even, separately.

b) Express your series with real numbers, sines and cosines.

c) At which points x does the series converge and to what value? Explain with statements of theorems.

2. Suppose that $f \in L^2(\mathbb{R}/2\pi\mathbb{Z})$ takes the form

$$f(\theta) = \sum_{n=1}^{\infty} a_n e^{in\theta}$$

Recall that if $z = re^{i\theta} = x + iy$,

$$F(z) = \sum_{n=1}^{\infty} r^n a_n e^{in\theta}$$

is a harmonic (and even analytic) function in $|z| < 1$.

a) Why does the series for $F(z)$ converge for $|z| < 1$?

b) Let $f_r(\theta) = F(re^{i\theta})$, the values of F on the circle of radius r. Calculate $\|f_r - f\|_2$ in terms r and a_n, and show that F takes on the boundary values in the sense that

$$\lim_{r \to 1^-} \|f_r - f\|_2 = 0$$

c) Evaluate the integral

$$\int\int_{|z|<1} |(\partial/\partial r)F(z)|^2 (1 - |z|)\, dx\, dy$$

in terms of the coefficients a_n. Explain at an appropriate point before, during or after the computation, why the integral is finite.

3. Fourier inversion on the Schwartz class $\mathcal{S}(\mathbb{R})$. (Approximate identity Lemma and Theorem 1 above.)

a) Recall that $C_0(\mathbb{R})$ is defined as the class of continuous functions on $\mathbb{R}$ that tend to zero at $\pm\infty$. Show that if $K \in L^1(\mathbb{R})$ and

$$\int_{\mathbb{R}} K(x)dx = 1; \qquad K_a(x) = \frac{1}{a}K(x/a), \quad a > 0$$

then

$$\lim_{a \to 0} f * K_a(x) = f(x)$$

for every $x \in \mathbb{R}$ and every $f \in C_0(\mathbb{R})$. Make use in your proof of the quantities

$$Q = \int_{\mathbb{R}} |K(x)|dx; \quad M = \max_{x\in\mathbb{R}} |f(x)|$$

and the modulus of continuity of f,

$$\omega(r) = \max_{x\in\mathbb{R};\ |y|\le r} |f(x+y) - f(x)|$$

b) Show that for every $f \in \mathcal{S}$, $f(0) = \dfrac{1}{2\pi}\displaystyle\int_{\mathbb{R}} \hat{f}(t)dt$. You may assume without proof that for every f and g in $\mathcal{S}$, $\hat{f}$ and $\hat{g}$ belong to $\mathcal{S}$ and

$$\int_{\mathbb{R}} f(y)\hat{g}(y)dy = \int_{\mathbb{R}} \hat{f}(t)g(t)dt$$

c) Deduce the Fourier inversion formula (formula for $f(x)$ in terms of $\hat{f}$) for $f \in \mathcal{S}$.

4. Fourier inversion formula on $L^2(\mathbb{R})$ (Proof of Theorem 3 and the proposition above.)

a) For $f \in L^2(\mathbb{R})$ and denote

$$s_N(x) = \frac{1}{2\pi}\int_{-N}^{N} \hat{f}(t)e^{ixt}dt$$

Explain why the integral defining $s_N(x)$ converges and why s_N is continuous.

b) Prove that if $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, then

$$(T_1 f)(t) = \int_{\mathbb{R}} f(x)e^{-itx}dx$$

following the three steps with *'s below.

You may assume that for any $f \in L^1 \cap L^2$, there is a sequence of functions $f_k \in \mathcal{S}$ such that $\|f - f_k\|_{L^1} + \|f - f_k\|_{L^2} \to 0$ as $k \to \infty$. Define

$$\varphi_k(t) = \int_{\mathbb{R}} f_k(x)e^{-itx}dx; \quad \varphi(t) = \int_{\mathbb{R}} f(x)e^{-itx}dx$$

* Show that $\varphi_k(t)$ tends to $\varphi(t)$ for each t as $k \to \infty$.

* Show that $\|\varphi_k - T_1 f\|_{L^2}$ tends to 0 as $k \to \infty$.

* Deduce that $\varphi(t) = (T_1 f)(t)$ (This equality holds in what sense?) Hint: Fatou's lemma leads to the fastest proof, but you may use other methods.

c) Deduce that

$$\lim_{N\to\infty} \int_{\mathbb{R}} |f(x) - s_N(x)|^2\, dx = 0$$

using the statement analogous to part (b) for T_2 and the other theorems on the page of theorems as necessary.

5. Poisson summation formula. Let $\varphi \in \mathcal{S}(\mathbb{R})$. Show that

$$\sum_{n\in\mathbb{Z}} \varphi(2\pi n) = \frac{1}{2\pi}\sum_{k\in\mathbb{Z}} \hat{\varphi}(k)$$

by calculating the Fourier series of

$$F(x) = \sum_{n\in\mathbb{Z}} \varphi(x - 2\pi n)$$

in two ways.

6. Recall that

$$P_y(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-y|\xi|}e^{ix\xi}\,d\xi = \frac{1}{\pi}\frac{y}{x^2+y^2}$$

satisfies for all $x \in \mathbb{R}$ and all $y > 0$,

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) P_y(x) = 0,$$

In other words, $P_y(x)$ is harmonic in the upper half-plane $\{(x, y) \in \mathbb{R}^2 : y > 0\}$ and for $f \in L^1(\mathbb{R})$,

$$u(x, y) = P_y * f(x)$$

is harmonic in the upper half plane $y > 0$.

If $f \in \mathcal{S}(\mathbb{R})$, use the Fourier transform to calculate

$$\int_0^{\infty}\int_{-\infty}^{\infty} |\nabla u(x, y)|^2 y\,dx\,dy$$

in terms of f. (Either before during or after the calculation, justify all the exchanges of integrals/differentiation/limits.)

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