Lecture 18: Regularity of L harmonic functions Part II

1 Finishing the proof

Last time we got as far as showing

$$\int_{B_r(x_0)} |\nabla u|^2 \leq 6 \int_{B_r(x_0)} |\nabla(u-v)|^2 + k' \left(\frac{r}{s}\right)^n \int_{B_s(x_0)} |\nabla v|^2, \tag{1}$$

and we can apply the lemma to get

$$\int_{B_r(x_0)} |\nabla u|^2 \leq \left(6\left(\frac{||A_{ij} - A_{ij}(x_0)||}{\lambda}\right)^2 + k'\left(\frac{r}{s}\right)^n\right) \int_{B_s(x_0)} |\nabla v|^2. \tag{2}$$

However, we need to eliminate v to use Morrey, so we need to replace the integral. Calculate

$$\begin{aligned} \int_{B_s(x_0)} |\nabla v|^2 &\leq 2\int_{B_s(x_0)} |\nabla u|^2 + 2\int_{B_s(x_0)} |\nabla(v-u)|^2 \qquad (3) \\ &\leq \left(2 + 2\left(\frac{||A_{ij} - A_{ij}(x_0)||}{\lambda}\right)^2\right) \int_{B_s(x_0)} |\nabla u|^2 \qquad (4) \end{aligned}$$

by the other part of our lemma. Substituting this back in we get

$$\int_{B_r(x_0)} |\nabla u|^2 \leq \left(6\left(\frac{||A_{ij} - A_{ij}(x_0)||}{\lambda}\right)^2 + k'\left(\frac{r}{s}\right)^n\right)\left(2 + 2\left(\frac{||A_{ij} - A_{ij}(x_0)||}{\lambda}\right)^2\right) \int_{B_s(x_0)} |\nabla u|^2. \tag{5}$$

By choosing s small we can get $||A_{ij} - A_{ij}(x_0)||$ small, and so,

$$\int_{B_r(x_0)} |\nabla u|^2 \leq \left(c_1 ||A_{ij} - A_{ij}(x_0)|| + c_2\left(\frac{r}{s}\right)^n\right) \int_{B_s(x_0)} |\nabla u|^2 \tag{6}$$