MATH 18.152 COURSE NOTES - CLASS MEETING # 11

18.152 Introduction to PDEs, Fall 2011 Professor: Jared Speck

Class Meeting # 11: The Method of Spherical Means

1. $1+3$ spacetime dimensions and the method of spherical means

We would now like to derive an analog of d'Alembert's formula in the physically relevant case of $1+3$ dimensions. As we will see, the analogous formula, known as Kirchhoff's formula, can be derived through the following steps.

- Given a solution $u(t,x)$ to the $1+3$ dimensional wave equation, we will define a spherical average of u centered at x. The average will depend on the averaging radius r.
- For fixed x, we will show that a slight modification of the average will solve the $1+1$ dimensional wave equation in the unknowns (t,r). With the help of our corollary to d'Alembert's formula, we will be able to find an explicit formula for this modified function.
- We will take a limit as the averaging goes to 0 in order to recover an expression for $u(t,x)$.

This procedure is known as the *method of spherical means*. The final result will be stated and proved as a theorem. Before proving the theorem, we will develop some preliminary estimates. We will use spherical coordinates $(r,\theta,\phi)\in[0,\infty)\times[0,\pi)\times[0,2\pi)$ on $\mathbb{R}^3$. Recall that if the spherical coordinates are centered at the Cartesian point (p^1,p^2,p^3), then the standard Cartesian coordinates (x^1,x^2,x^3) are connected to spherical coordinates by

$$x^1 = p^1 + r\sin\theta\cos\phi, \tag{1.0.1a}$$
$$x^2 = p^2 + r\sin\theta\sin\phi, \tag{1.0.1b}$$
$$x^3 = p^3 + r\cos\theta. \tag{1.0.1c}$$

Also recall that the integration measure associated to $B_r(0)$ is $d\sigma = r^2 d\omega$, where $d\omega \overset{\text{def}}{=} \sin\theta d\theta d\phi$. Here, ω represents the angular variables. We will abuse notation by using the symbol ω to denote both the angular coordinates (θ,ϕ), and alternatively as the corresponding point $(\sin\theta\cos\phi,\sin\theta\sin\phi,\cos\theta)\in\partial B_1(0)$.

Proposition 1.0.1 (Spherical averages). *Let $u(t,x)\in C^2([0,\infty)\times\mathbb{R}^3)$ be a solution to the $1+3$ dimensional global Cauchy problem*

$$-\partial_t^2 u(t,x) + \Delta u(t,x) = 0, \qquad (t,x)\in[0,\infty)\times\mathbb{R}^3, \tag{1.0.2a}$$
$$u(0,x) = f(x), \qquad x\in\mathbb{R}^3, \tag{1.0.2b}$$
$$\partial_t u(0,x) = g(x), \qquad x\in\mathbb{R}^3. \tag{1.0.2c}$$

For each $r > 0$, define the spherically averaged quantities

$$U(t,r;x) \overset{def}{=} \frac{1}{4\pi r^2}\int_{\partial B_r(x)} u(t,\sigma)\,d\sigma = \frac{1}{4\pi}\int_{\omega\in\partial B_1(0)} u(t,x+r\omega)\,d\omega, \tag{1.0.3a}$$

$$F(r;x) \overset{def}{=} \frac{1}{4\pi r^2}\int_{\partial B_r(x)} f(\sigma)\,d\sigma, \tag{1.0.3b}$$

$$G(r;x) \overset{def}{=} \frac{1}{4\pi r^2}\int_{\partial B_r(x)} g(\sigma)\,d\sigma, \tag{1.0.3c}$$

and their related modifications

$$\widetilde{U}(t,r;x) \overset{def}{=} rU(t,r;x), \tag{1.0.4a}$$

$$\widetilde{F}(r;x) \overset{def}{=} rF(r;x), \tag{1.0.4b}$$

$$\widetilde{G}(r;x) \overset{def}{=} rG(r;x). \tag{1.0.4c}$$

Then $\widetilde{U}(t,r;x) \in C^2([0,\infty)\times[0,\infty))$ is a solution to the following initial + boundary-value problem for the ***one-dimensional*** *wave equation:*

$$-\partial_t^2\widetilde{U}(t,r;x) + \partial_r^2\widetilde{U}(t,r;x) = 0, \qquad (t,r) \in [0,\infty)\times[0,\infty), \tag{1.0.5a}$$

$$\widetilde{U}(t,0;x) = 0, \qquad t \in [0,\infty), \tag{1.0.5b}$$

$$\widetilde{U}(0,r;x) = \widetilde{F}(r;x), \qquad r \in (0,\infty), \tag{1.0.5c}$$

$$\partial_t\widetilde{U}(0,r;x) = \widetilde{G}(r;x), \qquad r \in (0,\infty). \tag{1.0.5d}$$

Furthermore,

$$\lim_{r\to 0} U(t,r;x) = u(t,x). \tag{1.0.6}$$

Proof. Differentiating under the integral on the right-hand side of (1.0.3a), using the chain rule relation $\partial_r[u(t,x+r\omega)]\,d\omega = (\nabla u)(t,x+r\omega)\cdot\omega\,d\omega = \frac{1}{r^2}\nabla_{\hat{N}(\sigma)}u(t,\sigma)\,d\sigma$ (where $\hat{N}(\sigma)$ is the outward unit normal to $\partial B_r(x)$), and applying the divergence theorem, we compute that

$$\partial_r U = \frac{1}{4\pi r^2}\int_{\partial B_r(x)} \nabla_{\hat{N}(\sigma)}u(t,\sigma)\,d\sigma = \frac{1}{4\pi r^2}\int_{B_r(x)} \Delta_y u(t,y)\,d^3y. \tag{1.0.7}$$

We now derive a version of the fundamental theorem of calculus that will be used in our analysis below. If h is a continuous function on $\mathbb{R}^3$, then using spherical coordinates (ρ,ω) centered at the fixed point x, we have

(1.0.8)
$$\partial_r\int_{B_r(x)} h(y)\,d^3y = \partial_r\int_0^r\int_{\omega\in\partial B_1(0)} \rho^2 h(\rho,x+\rho\omega)\,d\omega d\rho = \int_{\omega\in\partial B_1(0)} r^2 h(r,x+r\omega)\,d\omega \overset{\mathrm{def}}{=} \int_{\partial B_r(x)} h(\sigma)\,d\sigma.$$

Multiplying both sides of (1.0.7) by r^2 and applying (1.0.8), we have that

$$\partial_r(r^2\partial_r U) = \frac{1}{4\pi}\partial_r \int_{B_r(x)} \Delta_y u(t,y)\,d^3y = \frac{1}{4\pi}\int_{\partial B_r(x)} \Delta u(t,\sigma)\,d\sigma. \tag{1.0.9}$$

Differentiating under the integral in (1.0.3a) and using (1.0.2a), we have that

$$\partial_t^2 U(t,r;x) = \frac{1}{4\pi r^2}\int_{\partial B_r(x)} \partial_t^2 u(t,\sigma)\,d\sigma = \frac{1}{4\pi r^2}\int_{\partial B_r(x)} \Delta u(t,\sigma)\,d\sigma. \tag{1.0.10}$$

Comparing (1.0.9) and (1.0.10), we see that

$$\partial_t^2 U(t,r;x) = \frac{1}{r^2}\partial_r(r^2\partial_r U) = \partial_r^2 U(t,r;x) + \frac{2}{r}\partial_r U(t,r;x). \tag{1.0.11}$$

Multiplying both sides of (1.0.11) by r and performing simple calculations, we see that

$$\partial_t^2\big[rU(t,r;x)\big] = \partial_r^2\big[rU(t,r;x)\big]. \tag{1.0.12}$$

We have thus shown that the PDE (1.0.5a) is verified by $\widetilde{U} \overset{\text{def}}{=} rU$.

Using (1.0.2b) - (1.0.2c) and definitions (1.0.3b) - (1.0.3c), it is easy to check that the initial conditions (1.0.5c) - (1.0.5d) hold. Note that you will have to differentiate under the integral in (1.0.3a) in order to show that (1.0.5d) holds.

The limit (1.0.6) follows easily from the right-hand side of (1.0.3a), since u is continuous.

Finally, the boundary condition (1.0.5b) then follows easily from multiplying (1.0.6) by r before taking the limit $r \to 0^+$.

□

Corollary 1.0.2 (**Representation formula for $\widetilde{U}(t,r;x)$**)**.** *Under the assumptions of Proposition 1.0.1, for $0 \le r \le t$, we have that*

$$\widetilde{U}(t,r;x) \overset{def}{=} rU(t,r;x) = \frac{1}{2}\Big(\widetilde{F}(r+t;x) - \widetilde{F}(r-t;x)\Big) + \frac{1}{2}\int_{\rho=-r+t}^{\rho=r+t} \widetilde{G}(\rho;x)\,d\rho. \tag{1.0.13}$$

Proof. (1.0.13) follows from (1.0.5a) - (1.0.5d) and the Corollary to d'Alembert's formula. □

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