Lecture 10

Interior $C^{2,\alpha}$ estimate for Newtonian potential (continued)

Proposition 1 *Consider* $B_1 = B_R(x_0) \subset B_{2R}(x_0) = B_2$*,* $f \in C^\alpha(B_{2R})$*, where* $0 < \alpha < 1$*. Let* $\omega(x) = \int_{B_2} \Gamma(x-y) f(y) dy$*, the Newtonian Potential of* f *in* B_2*. Then* $\omega \in C^{2,\alpha}(B_R(x_0))$ *and we have estimate*

$$\|D^2\omega\|_{0;B_R} + R^\alpha |D^2\omega|_{\alpha;B_R} \le C(\|f\|_{0;B_{2R}} + R^\alpha |f|_{\alpha;B_{2R}}),$$

where $C = C(n,\alpha)$ *is constant.*

Proof:(continued)

$$\begin{aligned}
(VI) &= (f(x) - f(\overline{x})) \int_{B_2 \setminus B_\delta(\xi)} D_{ij}\Gamma(x-y) dy \\
&\le |f|_{C^\alpha(x)} |x - \overline{x}|^\alpha | \int_{\partial(B_2 \setminus B_\delta(\xi))} D_i\Gamma(x-y)\nu_j ds_y| \\
&\le |f|_{C^\alpha(x)} \delta^\alpha (| \int_{\partial B_2} D_i\Gamma(x-y)\nu_j ds_y| + | \int_{\partial B_\delta(\xi)} D_i\Gamma(x-y)\nu_j ds_y|) \\
&\le c|f|_{C^\alpha(x)} \delta^\alpha (\int_{\partial B_2} \frac{1}{|x-y|^{n-1}} ds_y + \int_{\partial B_\delta(\xi)} \frac{1}{|x-y|^{n-1}} ds_y) \qquad (\frac{\delta}{2} \le \frac{1}{2}|y-\xi| \le |y-x|) \\
&\le c|f|_{C^\alpha(x)} (\delta^\alpha \frac{1}{R^{n-1}} n\omega_n (2R)^{n-1} + \frac{1}{(\delta/2)^{n-1}} n\omega_n (\delta)^{n-1}) \\
&\le c|f|_{C^\alpha(x)} \delta^\alpha.
\end{aligned}$$

$$\begin{aligned}
(V) &= \int_{B_2 \setminus B_\delta(\xi)} (D_{ij}\Gamma(\overline{x}-y) - D_{ij}\Gamma(x-y))(f(y) - f(\overline{x})) dy \\
&\le \int_{B_2 \setminus B_\delta(\xi)} |DD_{ij}\Gamma(\hat{x}-y)||x-\overline{x}||f|_{C^\alpha(\overline{x})} |\overline{x}-y|^\alpha dy \\
&\le c\delta |f|_{C^\alpha(\overline{x})} \int_{|y-\xi| \ge \delta} \frac{1}{|\hat{x}-y|^{n+1}} |\overline{x}-y|^\alpha dy
\end{aligned}$$

Since

$$|\overline{x}-y| \le |\overline{x}-\xi| + |\xi-y| \le \frac{\delta}{2} + |\xi-y| \le \frac{3}{2}|\xi-y|$$

and

$$\begin{aligned}
&|y-\xi| \le |y-\hat{x}| + |\hat{x}-\xi| \le |y-\hat{x}| + \frac{\delta}{2} \le |y-\hat{x}| + \frac{1}{2}|y-\xi| \\
&\Longrightarrow \frac{1}{2}|y-\xi| \le |y-\hat{x}|,
\end{aligned}$$

We thus get

$$
\begin{aligned}
(V) &\leq c\delta|f|_{C^\alpha(\overline{x})}\int_{|y-\xi|\geq\delta}\frac{dy}{|y-\xi|^{n+1-\alpha}} \leq c\delta|f|_{C^\alpha(\overline{x})}\int_\delta^{3R}\frac{1}{r^{n+1-\alpha}}r^{n-1}dr \\
&\leq c\delta|f|_{C^\alpha(\overline{x})}\int_\delta^{3R} r^{\alpha-2}dr \leq c\delta|f|_{C^\alpha(\overline{x})}\frac{1}{\alpha-1}((3R)^{\alpha-1}-\delta^{\alpha-1}) \\
&\leq \frac{c}{1-\alpha}|f|_{C^\alpha(\overline{x})}\delta^\alpha.
\end{aligned}
$$

Combine all the results, we have shown

$$|D_{ij}\omega(\overline{x}) - D_{ij}\omega(x)| \leq C(\frac{|f(x)|}{R^\alpha} + |f|_{C^\alpha(x)} + |f|_{C^\alpha(\overline{x})})|x-\overline{x}|^\alpha,$$

thus

$$R^\alpha\frac{|D_{ij}\omega(\overline{x}) - D_{ij}\omega(x)|}{|x-\overline{x}|^\alpha} \leq C(|f(x)| + R^\alpha|f|_{C^\alpha(x)} + R^\alpha|f|_{C^\alpha(\overline{x})}),$$

i.e.

$$R^\alpha|D^2\omega|_{\alpha;B_R} \leq C(\|f\|_{C^0;B_{2R}} + R^\alpha|f|_{C^\alpha(B_{2R})}).$$

Since we have already known that

$$|D^2\omega|_0 \leq c(\|f\|_{C^0} + R^\alpha|f|_\alpha),$$

we finally get

$$\|D^2\omega\|_{0;B_R} + R^\alpha|D^2\omega|_{\alpha;B_R} \leq C(\|f\|_{0;B_{2R}} + R^\alpha|f|_{\alpha;B_{2R}}). \qquad \blacksquare$$

Exercise:
1) Find a continuous function f s.t. $\Delta u = f$ does not have a C^2 solution.
2) Find $g \in C^1$ and $\Delta u = g$ but u is not $C^{2,1}$.

Interior $C^{2,\alpha}$ estimates for Poisson's equation.
Application: $u \in C^2(B_{2R}(x_0)), \Delta u = f, f \in C^\alpha(B_{2R}(x_0))$.

Theorem 1

$$\|u\|_{C^{2,\alpha}}(B_R) \leq \frac{C}{R^\alpha}(\|u\|_{C^0(B_{2R})} + \|f\|_{C^\alpha(B_{2R})}).$$

Proof: Since $\Delta(u - Nf) = 0$, where Nf is the Newtonian potential of f, thus from the $C^{2,\alpha}$ estimate of Nf we can get the $C^{2,\alpha}$ of u. $\blacksquare$

Theorem 2 *Let $u \in C^2(\Omega), \Delta u = f, f \in C^\alpha(\Omega)$, then for any $\Omega' \subset\subset \Omega$, we have*

$$\|u\|_{C^{2,\alpha}(\Omega')} \leq C(\|u\|_{C^0(\Omega)} + \|f\|_{C^\alpha(\Omega)}).$$

Proof: Take $x \in \Omega'$, choose R s.t.$B_{2R} \subset \Omega$, then

$$\begin{aligned} |u|_{C^{2,\alpha}(x)} \leq |u|_{C^{2,\alpha}(B_R(x))} &\leq \frac{C}{R^\alpha}(\|u\|_{C^0;B_{2R}} + |f|_{C^\alpha(B_{2R})}) \\ &\leq \frac{C}{R^\alpha}(\|u\|_{C^0;\Omega} + |f|_{C^\alpha(\Omega)}). \end{aligned}$$

Taking superior over all x and using previous estimate, we get

$$\|u\|_{C^{2,\alpha}(\Omega')} \leq C(\|u\|_{C^0(\Omega)} + \|f\|_{C^\alpha(\Omega)}). \qquad \blacksquare$$

Boundary estimate on Newtonian potential: $C^{2,\alpha}$ estimate up to the boundary for domain with flat boundary portion.

Suppose $B_R \subset B_{2R}$, with flat boundary portion $B_1^+ \subset B_2^+$.

Lemma 1 *Let $f \in C^\alpha(\overline{B_2^+})$, $\omega(x) = \int_{B_2^+} \Gamma(x-y)f(y)dy$ be the Newtonian potential of f in B_2^+. Then $\omega \in C^{2,alpha}(\overline{B_1^+})$ and*

$$|D^2\omega|_{0;B_1} + R^\alpha|D^2\omega|_{\alpha;B_1^+} \leq C(|f|_{0;B_2} + R^\alpha|f|_{\alpha;B_2^+}).$$

Proof: Examine the proof form last time. From C^2 estimate have

$$D_{ij}\omega(x) = \int_{B_2^+} D_{ij}\Gamma(x-y)(f(y)-f(x))dy - f(x)\int_{\partial B_2^+} D_i\Gamma(x-y)\nu_j(y)ds_y.$$

(Note $\int_{\partial B_2^+} D_i\Gamma(x-y)\nu_j(y)ds_y = \int_{\partial B_2^+} D_j\Gamma(x-y)\nu_i(y)ds_y$.)

So if either $i \neq n$ or $j \neq n$, the integral on the lower boundary portion of B_2^+ vanishes. (Since $\nu = (0, 0, \cdots, -1)$.)

If $i = j = n$, then

$$\begin{aligned} D_{nn}\omega &= f(x)\int_{\partial B_2^+} (D_n\Gamma(x-y) - D_n\Gamma(\overline{x}-y))(-1)d\sigma \\ &\leq |f(x)|\int_{\partial B_2^+} |DD_n\Gamma(\hat{x}-y)||x-\overline{x}|d\sigma \\ &\leq |f(x)|\delta\int_{\partial B_2^+} \frac{1}{|\hat{x}-y|^n}d\sigma \\ &\leq |f(x)|\delta\frac{1}{R^n}n\omega R^{n-1} \\ &\leq c|f(x)|\delta^\alpha. \end{aligned}$$

Since we know $\Delta\omega = f$, thus $\omega_{nn} = f - \omega_{11} - \omega_{22} - \cdots$, so we see that $\omega_{nn} \in C^\alpha$, and we can get estimate for ω_{nn} from estimates for $\omega_{ii}, i < n$. $\blacksquare$