Lecture 24

May 13th, 2004

Our motivation for this last lecture in the course is to show a result using our regularity theory which is otherwise unprovable using classical techniques. This is the previous Theorem, and in particular the case $p \neq 2$ (which we haven't yet done) namely N is a continuous map $L^p(\Omega)$ to $W^{2,p}(\Omega)$. Classical methods give at best $W^{1,p}(\Omega)$. For that we introduce the

Calderon-Zygmund Decomposition Technique: Cube decomposition

$\mathcal{L}$et K_0 be an $n-$dimensional cube in $\mathbb{R}^n$, $f \geq 0$ integrable and finally fix $t > 0$ such that

$$\int_{K_0} f \leq t|K_0| \equiv t\mathrm{Vol}(K_0), \quad \text{that is } ⨍_{K_0} f \leq t.$$

Next bisect K_0 into 2^n equal (in volume) subcubes. Let $\mathcal{S}$ be the collection of those subcubes K for which $⨍_K f > t$. I.e the subcubes where f is highly concentrated. On *each* of the remaining subcubes (those not in $\mathcal{S}$) repeat the same procedure, i.e bisect each one into 2^n sub-subcubes and add those where f is highly concentrated to $\mathcal{S}$, bisect the rest et ceterà… Now for any $K \in \mathcal{S}$ denote by $\tilde{K}$ its immediate predecessor. Since $K \in \mathcal{S}$ while $\tilde{K} \notin \mathcal{S}$

$$t < \frac{1}{\mathrm{Vol}(K)} \int_K f < \frac{1}{\mathrm{Vol}(K)} \int_{\tilde{K}} f = \frac{\mathrm{Vol}(\tilde{K})}{\mathrm{Vol}(K)} \cdot \frac{1}{\mathrm{Vol}(\tilde{K})} \int_{\tilde{K}} f \leq 2^n t.$$

In summary $\forall\ K \in \mathcal{S} \quad t < ⨍_K f \leq 2^n t$. Denote $F := \bigcup_{K\in\mathcal{S}} K$, $G := K_0 \setminus F \equiv F^C = \bigcap_{K\in\mathcal{S}} K^C$. We see each point in G lies in infinitely many nested cubes with bounded concentration of f with diameters converging to 0: $⨍_{K_i} f \leq t$ with $\mathrm{Vol}(K_i) \longrightarrow 0$. By Lebesgue's Theorem on differentiation the LHS $\longrightarrow f$ $\lambda-$a.e (λ denotes Lebesgue's measure on $\mathbb{R}^n$), i.e $f \leq t$ a.e ($\equiv \lambda-$a.e) on G. In summary, on F we have an average estimate, on G we have a pointwise estimate.

The promised proof

$\mathcal{W}$e restate the desired result whose proof we gave for $p = 2$.

Theorem. *Let $f \in L^p(\Omega)$ for some $1 < p < \infty$ and let $\omega = Nf$ be the Newtonian Potential of f. Then $\omega \in W^{2,p}(\Omega)$ and $\Delta w = f$ a.e. and*

$$||D^2 w||_{L^p(\Omega)} \le c(n,p,\Omega) \cdot ||f||_{L^p(\Omega)}.$$

For $p = 2$ we have even

$$\int_{\mathbb{R}^n} |D^2\omega|^2 = \int_\Omega f^2.$$

Proof. Define an operator $T : L^2(\Omega) \longrightarrow L^2(\Omega)$, $Tf = \mathrm{D}_{ij}Nf$. Last time we showed $||\mathrm{D}_{ij}Nf||_{L^2(\Omega)} = ||Tf||_{L^2(\Omega)} = ||f||_{L^2(\Omega)}$. In other words T is strong $(2,2)$ and therefore automatically weak $(2,2)$ i.e

$$\mu_{Tf} \le \Big(\frac{||f||_{L^2(\Omega)}}{t}\Big)^2 \tag{1}$$

by the Proposition in the previous lecture. If we will now be able to bound it with $\dfrac{||f||_{L^1(O)}}{t}$, the Interpolation Theorem will then provide the desired bound on $\mathrm{D}^2\omega$ for all $1 < p < 2$. By duality $2 < p < \infty$ will then be taken care of as well (to be made precise). So we

Claim. *T is weak* $(1,1)$ *i.e*

$$\forall\, f \in L^2(\Omega) \cap L^1(\Omega) \qquad \mu_{Tf}(t) \le C\frac{||f||_{L^1(\Omega)}}{t}, \quad \forall\, t > 0.$$

Proof. Extend f trivially outside Ω (i.e so the extension vanishes on $\mathbb{R}^n \setminus \Omega$), and given any fixed $t > 0$ take a large enough cube K_0 containing Ω such that

$$\fint_{K_0} |f| = \frac{1}{\mathrm{Vol}(K_0)} \int_{K_0} |f| \leq t.$$

The Cube Decomposition furnishes a countable number of cubes $\{K_l\}$ such that on each $t < \fint_{K_l} |f| \ \leq \ 2^n t$ and in addition $|f| \leq t$ a.e on $G := K_0 \setminus \bigcup_l K_l$. Split $f = b + g$ into bad, good parts by letting $g(x) := \begin{cases} f(x) & \text{on } G \\ \fint_{K_l} f & \text{on } K_l \end{cases}$ i.e f could be oscillating on K_l, instead we just replace it there by its average value therein. Then let $b := f - g$, the bad or highly oscillating part. Note: $|g| \leq 2^n t$ a.e, $b(x) = 0$ on G and $\int_{K_l} b = 0$.

We have now $Tf = Tb + Tg$. And as in the Interpolation Theorem of the previous lecture

$$\mu_{Tf}(t) \leq \mu_{Tb}(t/2) + \mu_{Tg}(t/2).$$

We would like to bound this with the $L^1(\Omega)$ norm of f. We divide the computation into 3 parts.

$L^1(\Omega)$ **estimate for** $\mu_{Tg}(t/2)$**.** Using (1) on the good part we have

$$\mu_{Tg}(t/2) \leq \Big(\frac{||g||_{L^2(\Omega)}}{t/2}\Big)^2$$

$$\leq \frac{\int_{K_0} g^2}{(t/2)^2}$$

and since $g/(2^n t) \leq 1$, $\big(g/(2^n t)\big)^2 \leq |g|/(2^n t)$ or $\big(g/t\big)^2 \leq 2^n |g|/t$ from which

$$\leq \frac{2^{n+2}}{t} \int_{K_0} |g|$$

$$= \frac{2^{n+2}}{t} \int_G + \int_{\cup_l K_l} |g|$$

$$= \frac{2^{n+2}}{t} \int_G |f| + \int_{\cup_l K_l} \big(\fint_{K_l} |f| \big)$$

$$= \frac{2^{n+2}}{t}\int_{\Omega}|f| = \frac{2^{n+2}}{t}||f||_{L^1(\Omega)}.$$

We have not used so far any properties of T. On the bad part we will, and we will work with the kernel of the Newtonian Potential, in just a moment.

$L^1(K_0 \setminus \bigcup_l B_l)$ **estimate for** Tb**.** Let $\bar{y}$ be the center of the subcube K_l. Let $B_l := B(\bar{y}, \delta)$ which strictly contains K_l. The diameter of K_l is $\delta := \text{diam}(\Omega)\frac{\sqrt{n}}{2^r}$ if it belongs to the r^{th} subdivision.

We content ourselves with bounding only the L^1 norm of Tb on $K_0 \setminus \bigcup_l B_l$ since by part I) of the Proposition of Lecture 23 (with $p = 1$) that will bound the distribution function μ_{Tb} itself.

Write $b_l := \mathbb{I}_{K_l}$, the characteristic function defined in Lecture 18. $\quad b = \sum_{l=1}^{\infty} b_l$. The advantage of this splitting is that each term is compactly supported unlike b itself. Fix some $l \in \mathbb{N}$ and approximate b_l by smooth functions $\{b_l^{(m)}\}_{m=0}^{\infty} \subseteq \mathcal{C}_0^{\infty}(K_l)$. By varying each with a constant one can make sure for each $n \in \mathbb{N}$ $\int_{K_l} b_l^{(m)} = \int_{K_l} b_l = 0$.

If $x \in K_l$,

$$T(b_l^{(m)})(x) = \int_{K_l} \text{D}_{ij}\Gamma(x-y)b_l^{(m)}(y)d\mathbf{y}$$

$$= \int_{K_l} \left[\text{D}_{ij}\Gamma(x-y) - \text{D}_{ij}\Gamma(x-\bar{y})\right]b_l^{(m)}(y)d\mathbf{y}$$

by the zero average $b_l^{(m)}$.

Computation.

$$\left|Tb_l^{(m)}(x)\right| \le \; c \cdot \delta \cdot \frac{1}{[dist(x, K_l)]^{n+1}} \int_{K_l} |b_l^{(m)}(y)|d\mathbf{y}.$$

Proof. Using the above equation in conjunction with the Mean Value Theorem of Calculus there exists $y_0 \in K_l$ (and $|y - y_0| \le \delta \quad \forall\, y \in K_l$) such that

$$\begin{aligned}
\big|Tb_l^{(m)}(x)\big| &= \Big|\int_{K_l} \mathrm{DD}_{ij}\Gamma(x-y_0)\cdot(y-y_0)b_l^{(m)}(y)d\mathbf{y}\Big| \\
&\le \int_{K_l} |\mathrm{DD}_{ij}\Gamma(x-y_0)|\cdot|y-y_0||b_l^{(m)}(y)|d\mathbf{y} \\
&\le c\delta\int_{K_l}\frac{1}{|x-y_0|^{n+1}}|b_l^{(m)}(y)|d\mathbf{y} \\
&\le c\delta\frac{1}{[\mathrm{dist}(x,K_l)]^{n+1}}\int_{K_l}|b_l^{(m)}(y)|d\mathbf{y}. \qquad \blacksquare
\end{aligned}$$

This now helps us evaluate the L^1 norm

$$\int_{K_0\setminus B_l}\big|Tb_l^{(m)}\big| \le\ c\cdot\delta\int_{|x-\bar{y}|\ge\delta}\frac{1}{[\mathrm{dist}(x,K_l)]^{n+1}}d\mathbf{x}\cdot\Big(\int_{K_l}|b_l^{(m)}(y)|d\mathbf{y}\Big).$$

Note there is some $\tilde{y}$ with $\mathrm{dist}(x,K_l)=|x-\tilde{y}|$ and then $|x-\bar{y}|\le|x-\tilde{y}|+|\tilde{y}-y_0|\le 2\mathrm{dist}(x,K_l)$

$$\begin{aligned}
&\le 2^{n+1}\ c\cdot\delta\int_{|x-\bar{y}|\ge\delta}\frac{1}{|x-\bar{y}|^{n+1}}d\mathbf{x}\cdot\Big(\int_{K_l}|b_l^{(m)}(y)|d\mathbf{y}\Big) \\
&= c'\int_{K_l}|b_l^{(m)}(y)|d\mathbf{y}.
\end{aligned}$$

Let $m\to\infty$ in the above to get

$$\int_{K_0\setminus B_l}\big|Tb_l\big|\le c'\int_{K_l}|b_l(y)|d\mathbf{y}$$

i.e we have taken care of things (have $L^1(\Omega)$ estimates there) on $K_0\setminus\bigcup_l B_l$, as can be seen by summing (the b_l's have disjoint supports so $|b|=\sum_l|b_l|$)

$$||Tb||_{L^1(K_0\setminus\bigcup_l B_l)} = \int_{K_0\setminus\bigcup_l B_l} |Tb| \le \sum_{l=1}^{\infty} \int_{K_0\setminus B_l} |Tb_l|$$

$$\le \sum_{l=1}^{\infty} c' \int_{K_l} |b_l|$$

$$\le c' \int_{\bigcup_l B_l} |b| = c' \int_{\bigcup_l B_l} |f| = c'||f||_{L^1(\Omega)}.$$

$L^1(\bigcup_l B_l)$ **estimates for** $\mu_{Tb}(t/2)$**.**

$$\mu_{Tb}(t/2) = \big|\{x \in \Omega : \ Tb(x) > t/2\}\big| \le \big|\{\alpha \in K_0 \setminus \bigcup_l B_l : \ |Tb| > t/2\}\big| + \big|\bigcup_l B_l\big|.$$

The first term is taken care of (by applying part I of the Proposition in Lecture 23 with $p = 1$ to the estimate above for $||Tb||_{L^1(K_0\setminus\bigcup_l B_l)}$). For the second, there exists some constant c such that $\big|\bigcup_l B_l\big| \le c\big|\bigcup_l K_l\big|$ by the geometry of cubes and balls. Now the K_l were chosen with

$$t < \fint_{K_l} |f|,$$

hence

$$\mathrm{Vol}(K_l) < \frac{1}{t}||f||_{L^1(K_l)}.$$

Altogether

$$\mu_{Tb}(t/2) \le \frac{c}{t}||f||_{L^1(\Omega)}.$$

Combining the above 3 parts

$$\forall\ f \in L^2(\Omega) \qquad \mu_{Tf}(t) \le \mu_{Tb}(t/2) + \mu_{Tg}(t/2). \le \frac{c}{t}||f||_{L^1(\Omega)}. + \frac{2^{n+1}}{t}||f||_{L^1(\Omega)}, \quad \forall\ t > 0.$$

That is T is weak $(1, 1)$ proving the Claim. ■

Thus by the Marcinkiewicz Interpolation Theorem (MIT) exists c depending on the above constants, i.e on $n, p, \mathrm{diam}(\Omega)$, satisfying

$$\forall\ f \in L^2(\Omega) \qquad ||Tf||_{L^p(\Omega)} \leq c||f||_{L^p(\Omega)}, \quad 1 < p < 2\ ! \tag{2}$$

From the proof of the MIT c blows up as p approaches either of the endpoints. We mention without proof that as stronger version of the MIT states that if T is strong (r, r) and/or strong (q, q) then the constant does not blow-up at r and/or q. *Therefore we have in fact* $1 < p \leq 2$ *in* (2). As a matter of fact we do not even need to invoke this stronger Theorem since we have done the case $p = 2$ independently (with constant $= 1$!) in the previous lecture.

Yet another idea would be to prove (2) for some one value p greater than 2, and apply the MIT to get (2) for $p = 2$ as an intermediate value in the interval $(1, p)$! This will also conclude the proof of our Theorem as p will be arbitrary.

To that end we use the so called *Duality Method.* Let $p > 2$ be arbitrary. $(L^p(\Omega))^\star = L^q(\Omega)$ with $1 = \frac{1}{q} + \frac{1}{p}$. By the definition of the dual space (p. 3 of Lecture 17)

$$\begin{aligned}
||Tf||_{L^p(\Omega)} = \sup_{\substack{g \in L^q(\Omega) \\ ||g||_{L^q(O)}=1}} \int_\Omega Tf \cdot g &= \sup_{\substack{g \in L^q(\Omega) \\ ||g||_{L^q(O)}=1}} \int_\Omega \mathrm{D}_{ij}\omega \cdot g \\
&= \sup_{\substack{g \in L^q(\Omega) \\ ||g||_{L^q(O)}=1}} \int_\Omega \omega \cdot \mathrm{D}_{ij} g \\
&= \sup_{\substack{g \in L^q(\Omega) \\ ||g||_{L^q(O)}=1}} \int_\Omega \Big(\int_\Omega \Gamma(x-y) f(y) d\mathbf{y} \Big) \mathrm{D}_{ij} g(x) d\mathbf{x} \\
&= \sup_{\substack{g \in L^q(\Omega) \\ ||g||_{L^q(O)}=1}} \int_\Omega \Big(\int_\Omega \Gamma(x-y) \mathrm{D}_{ij} g(x) f(y) d\mathbf{x} \Big) d\mathbf{y} \\
&= \sup_{\substack{g \in L^q(\Omega) \\ ||g||_{L^q(O)}=1}} \int_\Omega \Big(\int_\Omega \mathrm{D}_{ij}\Gamma(x-y) g(x) d\mathbf{x} \Big) f(y) d\mathbf{y} \\
&= \sup_{\substack{g \in L^q(\Omega) \\ ||g||_{L^q(O)}=1}} \int_\Omega Tg \cdot f
\end{aligned}$$

$$\underset{\text{Hölder's Ineq.}}{\leq} \sup_{\substack{g \in L^q(\Omega) \\ ||g||_{L^q(O)}=1}} ||f||_{L^p(\Omega)} \cdot ||Tg||_{L^q(\Omega)}$$

$$\leq C \sup_{\substack{g \in L^q(\Omega) \\ ||g||_{L^q(O)}=1}} ||f||_{L^p(\Omega)} \cdot ||g||_{L^q(\Omega)}$$

$$= C||f||_{L^p(\Omega)} \cdot 1.$$

As we wished: T is strong (p, p). In the last inequality we simply used the fact that $1 < q < 2$ is in the range we have already taken care of.

In summary we have shown: If $f \in \mathcal{C}_0^\infty(\Omega)$, $\omega := Nf$ then $\Delta\omega = f$ and $||\mathrm{D}^2\omega||_{L^p(\Omega)} \leq c||f||_{L^p(\Omega)}$ for any $1 < p < \infty$. Now identically to how we finished the proof of the last Theorem in the previous lecture we extend this to all functions in $L^p(\Omega)$ by approximating and subsequently taking limits and making use of Young's Inequality (Lecture 23). ■

Our work can be rephrased

Corollary. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain and assume $u \in W_0^{2,p}(\Omega)$ for some $1 < p < \infty$. Then*

$$||D^2u||_{L^p(\Omega)} \leq c(n, p, \Omega) \cdot ||\Delta u||_{L^p(\Omega)}.$$

For $p = 2$

$$||D^2u||_{L^2(\Omega)} = ||\Delta u||_{L^p(\Omega)}.$$

Proof. $u - N(\Delta u)$ satisfies Laplace's equation $\Delta(u - N(\Delta u)) = 0$ a.e. In other words $u - N(\Delta u)$ is a harmonic function with compact support in $\mathbb{R}^n$ hence vanishes identically. Hence $u = N\Delta u$ and renaming $f := \Delta u, \omega := u$ gives the inequality from our above Theorem. ■

This is quite remarkable as it tells us that the whole Hessian, $\binom{n}{2}$ functions, can be bounded simply in terms of the sum of its n diagonal terms — its trace.

Theorem. *Let* $L := a^{ij}D_{ij} + b^iD_i + c$ *and let* $\Omega \subseteq \mathbb{R}^n$ *be a bounded domain. Assume* $u \in W^{2,p}(\Omega)$ *for some* $1 < p < \infty$ *satisfies* $Lu = f$ *a.e. Assume*

- L *strictly elliptic with* $(a^{ij}) > \gamma \cdot I, \;\; \gamma > 0$
- $a^{ij} \in \mathcal{C}^0(\Omega)$
- $b^i, c \in L^\infty(\Omega)$
- $f \in L^p(\Omega)$.

Then $\forall\, \Omega' \Subset \Omega$ *holds*

$$||u||_{W^{2,p}(\Omega')} \leq c \cdot \big(||u||_{L^p(\Omega)} + ||f||_{L^p(\Omega)}\big).$$

Proof. We know this for $L = \Delta$, and therefore for any constant coefficients operator satisfying the above by Lecture 12. Then perturbing the coefficients and proceeding just like in the Schauder case works, as in Lecture 13, works. ■

Assuming $\mathcal{C}^{1,1}$ boundary, these estimates can be extended to hold globally on all of Ω, as in done in Lecture 14.

♣