Lecture 11

Review of Green's functions.

$G : \Omega \times \Omega \longrightarrow \mathbb{R}$.

Given $x \in \Omega$, let $h_x(y) : \Omega \longrightarrow \mathbb{R}$ be s.t. $\Delta_y h_x(y) = 0$ and $h_x(y) = -\Gamma(|x-y|)$ for $y \in \partial\Omega$.

By definition, $G(x,y) = \Gamma(|x-y|) + h_x(y)$.

If Green's function exists, then for $u \in C^1(\overline{\Omega}) \cap C^2(\Omega), y \in \Omega$, we have

$$u(y) = \int_{\partial\Omega} u(x) \frac{\partial G(x,y)}{\partial \nu} d\sigma + \int_{\Omega} G(x,y)\Delta u(x) dx.$$

Thus we can see:
If $u = 0$ on $\partial\Omega$, then $u(y) = \int_\Omega G(x,y)\Delta u(x)dx = G * \Delta u$.

(Compare) By Green's formula, we have
If $u \in C_c^2(\mathbb{R}^n)$, then $u(y) = \Gamma * \Delta u$.

Proposition 1 *a)* $G(x,y) = G(y,x)$*;*
b) $G(x,y) < 0$*, for* $x, y \in \Omega, x \neq y$*.*
c) $\int_\Omega G(x,y)f(y)dy \to 0$ *as* $x \to \partial\Omega$*, where* f *is bounded and integrable.*

Proof of c): From definition, $G(x,y) = 0$ if $x \in \Omega, y \in \partial\Omega$.

By a), $G(x,y) = 0$ for $y \in \Omega, x \in \partial\Omega$.

Thus $G : \overline{\Omega} \times \overline{\Omega} - \{diag\} \longrightarrow \mathbb{R}$.

$$\begin{aligned} |\int_\Omega |G(x,y)f(y)|dy &\leq \|f\|_{L^\infty} \int_\Omega |G(x,y)|dy \\ &\leq \|f\|_{L^\infty} \int_\Omega \frac{C}{|x-y|^{n-2}} dy \\ &\leq C\|f\|_{L^\infty}. \end{aligned}$$

By dominate convergence, we can change limit and integral. ■

Example. Green's function for $\mathbb{R}^n_+$

Given $y = (y^1, \cdots, y^n)$, let $y^* = (y^1, \cdots, y^{n-1}, -y^n)$.

It is easy to check that $G(x) = \Gamma(x-y) - \Gamma(x-y^*) = \Gamma(x-y) - \Gamma(x^*-y)$ is Green's function for $\mathbb{R}^n_+$:

- $h_x(y) = G(x,y) - \Gamma(x-y)$ is harmonic in Ω;
- $G(x,y) = 0$ on $\partial\Omega$.

Review of Schwartz reflection.

First we go back to harmonic functions.

Theorem 1 *A $C^0(\Omega)$ function u is harmonic if and only if for every ball $B_R(y) \subset\subset \Omega$, we have*

$$u(y) = \frac{1}{n\omega_n R^{n-1}} \int_{\partial B} u ds.$$

Proof: $\Longrightarrow$ is just mean value theorem.

$\Longleftarrow$: Use the Poisson kernel: Given any Ball $B_R(y) \subset \Omega$, Define

$$h(x) = \begin{cases} \frac{R^2-|x^2|}{n\omega_n R} \int_{\partial B} \frac{u(y)}{|x-y|^n} ds & , \quad x \in B_R, \\ u(x) & , \quad x \in \partial B. \end{cases}$$

Then $h \in C^2(B_R) \cap C^0(\overline{B_R})$ and satisfies $\Delta u = 0$. So h satisfies the mean value property. Therefore $u - h$ satisfies the mean value property and $u = h$ on ∂B_R.

But recall the uniqueness theorem for solutions of Poisson's equation – we only used the mean value property. Therefore $u = h$, so u is harmonic. ■

Now suppose $\Omega^+ \subset \mathbb{R}^n_+$, $T = \overline{\Omega^+} \cap \partial\mathbb{R}^n_+$ is a domain in $\partial\mathbb{R}^n_+$. Let $\Omega^- = (\Omega^+)^*$, i.e.

$$\Omega^- = \{(x_1, \cdots, x_n) \in \mathbb{R}^n | (x_1, \cdots, -x_n) \in \Omega^+\}.$$

Suppose we have u harmonic in Ω^+, $u \in C^0(\Omega^+ \cup T)$, and $u = 0$ on T. Define

$$u(x_1, \cdots, x_n) = \begin{cases} u(x_1, \cdots, x_n) & , \quad x\Omega^+ \cup T, \\ u(x_1, \cdots, -x_n) & , \quad x \in \Omega^-. \end{cases}$$

Theorem 2 *The function u defined above is harmonic in $\Omega^+ \cup T \cup \Omega^-$.*

Proof: Obviously u is in $C^0\Omega^+ \cup T \cup \Omega^-$.

If one examines the above proof, one only requires that for each point $y \in \Omega$, $\exists R > 0$ so that mean value property holds in $B_r(y), r < R$. Also remember in the proof of maximum principle, we assumed that the function has a interior max, then use mean value theorem in small ball around this point.

Certainly here we have this property in $\Omega^+ \cup \Omega^-$, and on T if follows from the definition of u, $\int_{\partial B_R(x \in T)} u = 0$. ■

$C^{2,\alpha}$ boundary estimate for Poisson's equation with flat boundary portion.

Theorem 3 *Let $u \in C^2(B_2^+) \cap C^0(\overline{B_2^+}), f \in C^\alpha(B_2^+)$, and $\Delta u = f$ in B_2^+, $u = 0$ on T. Then $u \in C^{2,\alpha}(B_1^+)$ and*

$$\|u\|_{C^{2,\alpha}(B_1^+)} \leq C(\|u\|_{C^0(B_2^+)} + \|f\|_{C^\alpha(B_2^+)}).$$

Proof: Reflect f with respect to T, i.e.

$$f^*(x) = f^*(x_1, \cdots, x_n) = \begin{cases} f(x_1, \cdots, x_n) & , \quad x_n \geq 0, \\ f(x_1, \cdots, -x_n) & , \quad x_n \leq 0. \end{cases}$$

Let $D = B_2^+ \cup B_2^- \cup (B_2 \cap T)$, then $f^* \in C^\alpha(\overline{D})$ and $\|f\|_{C^\alpha(D)} \leq 2\|f\|_{C^\alpha(B_2^+)}$. Let $G(x,y)$ be the Green's function of upper half space. Define

$$\begin{aligned} \omega(x) &= \int_{B_2^+} G(x,y) f(y) dy \\ &= \int_{B_2^+} (\Gamma(x-y) - \Gamma(x-y^*)) f(y) dy \\ &= \int_{B_2^+} (\Gamma(x-y) - \Gamma(x^*-y)) f(y) dy \\ &= \int_{B_2^+} \Gamma(x-y) f(y) dy - \int_{B_2^-} \Gamma(x-y) f^*(y) dy. \end{aligned}$$

Then $\Delta\omega = f$. It's easy to check that $\omega(x) = 0$ on T. Thus

$$\int_{B_2^-} \Gamma(x-y) f^*(y) dy = \int_D \Gamma(x-y) f^*(y) dy - \int_{B_2^+} \Gamma(x-y) f(y) dy,$$

so

$$\omega(x) = 2\int_{B_2^+} \Gamma(x-y) f(y) dy - \int_D \Gamma(x-y) f^*(y) dy.$$

We did estimates for the first term earlier. For the second term, think of $B_1^+ \subset D$ and just use interior estimates from last week. We thus get

$$\|\omega\|_{C^{2,\alpha}(B_1^+)} \leq C\|f\|_{C^{0,\alpha}(B_2^+)}.$$

Let $v = u - \omega$ in B_2^+, then on B_2^+ we have $\Delta v = \Delta u - \Delta\omega = f - f = 0$ and $v = 0$ on T.

We may reflect v, then by Schwartz reflection we know that v^* is harmonic in D. Now use the interior estimates for harmonic functions, we get

$$\|v\|_{C^{2,\alpha}(B_1^+)} \leq C\|v^*\|_{C^0(D)} \leq 2\|v\|_{C^0(D)}.$$

So

$$\|u\|_{C^{2,\alpha}(B_1^+)} \leq \|v\|_{C^{2,\alpha}(B_1^+)} + \|\omega\|_{C^{2,\alpha}(B_1^+)} \leq C(\|u\|_{C^0(B_2^+)} + \|f\|_{C^\alpha(B_2^+)}). \qquad \blacksquare$$

Application: Global $C^{2,\alpha}$ Regularity Theorem for Dirichlet problem in a ball with zero boundary data.

Theorem 4 *Suppose B is a ball in $\mathbb{R}^n$, $u \in C^2(B) \cap C^0(\overline{B})$, $f \in C^\alpha(\overline{B})$, $\Delta u = f$ in B and $u = 0$ on ∂B. Then $u \in C^{2,\alpha}(\overline{B})$.*

Proof: By dilation and translation, we can assume $B = B_{1/2}(0, \cdots, 0, \frac{1}{2})$.

Look at the inversion $x \to Ix = \frac{x}{|x|^2}$, then the ball B is mapped to a half space $B^* = \{x | x_n \geq 1\}$ while ∂B is mapped onto $\partial B^* = \{x_n = 1\}$.

The Kelvin Transform of u is $v(x) = |x|^{2-n} u(\frac{x}{|x|^2}) \in C^2(B^*) \cap C^0(\overline{B^*})$ and we have

$$\Delta_y v(y) = |y|^{-n-2} \Delta_x u(x) = |y|^{-n-2} f(\frac{y}{|y|^2}) \in C^\alpha(B^*).$$

By the previous theorem, $u \in C^{2,\alpha}$ up to the boundary.

By rotation, we could do this for any boundary point, so $u \in C^{2,\alpha}$. ■

Corollary 1 *Suppose $\varphi \in C^{2,\alpha}(\overline{B}), f \in C^\alpha(\overline{B})$. Then the Dirichlet problem*

$$\begin{cases} \Delta u = f & , \quad x \in B, \\ u = \varphi & , \quad x \in \partial B. \end{cases}$$

is uniquely solvable for $u \in C^{2,\alpha}(\overline{B})$.

Proof: The existence of u comes from Perron's method.

Since $\Delta\varphi \in C^\alpha(\overline{B})$, so let v be the unique solution of $\Delta v = f - \Delta\varphi$ in B with $v = 0$ on ∂B. Then $v \in C^(B) \cap C^0(\partial\overline{B})$. By above result, $v \in C^{2,\alpha}(\overline{B})$.

But $u - \varphi$ solves the problem also: $\Delta(u - \varphi) = \Delta u - \Delta\varphi = f - \Delta\varphi$ in $\overline{B}; u - \varphi = 0$ on $\partial\overline{B}$. By uniqueness, $v = u - \varphi$. So $u \in C^{2,\alpha}(\overline{B})$. ■