Lecture 6

Weak maximum principle for linear elliptic operators

Now we consider the more general differential operators

$$L = a^{ij}(x)D_{ij} + b^i(x)D_i + c(x),$$

i.e., for any C^2 function u,

$$Lu = a^{ij}(x)\frac{\partial^2 u(x)}{\partial x^i \partial x^j} + b^i(x)\frac{\partial u(x)}{\partial x^i} + c(x)u(x),$$

where a^{ij}, b^i, c are bounded functions.

Definition 1 *Suppose L is like above.*
1. If $\exists \lambda(x) > 0$ s.t. $(a^{ij}(x)) > \lambda(x)I$, then L is **elliptic.**
2. If $\exists \lambda(x) > \lambda_0 > 0$ s.t. $(a^{ij}(x)) > \lambda(x)I$, then L is **strictly elliptic.**
3. If $\exists \infty > \Lambda > \lambda_0 > 0$ s.t. $\Lambda I > (a^{ij}(x)) > \lambda_0 I$, then L is **uniformly elliptic.**

Theorem 1 *Suppose L is elliptic in bounded domain Ω, $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$, $Lu \geq 0, c(x) \equiv 0$ in Ω, then*

$$\sup_{\Omega} u = \sup_{\partial\Omega} u.$$

If $Lu \leq 0$ instead, then

$$\inf_{\Omega} u = \inf_{\partial\Omega} u.$$

Proof: Assume $x_0 \in \Omega$ s.t. $u(x_0) = \sup_\Omega u$, then $(D_{ij}u(x_0)) \leq 0, D_i u(x_0) = 0$, so we get

$$Lu(x_0) = a^{ij}D_{ij}u(x_0) \leq 0.$$

If $Lu > 0$, then we have already get a contradiction. So the theorem is true for this simple case.

Now we turn to the general case $Lu \geq 0$. Without loss of generality, we can assume $a^{11} > 0$. Let $v = e^{rx^1}$ for some constant r, then

$$v_i = re^{rx^1}\delta_{1i}, \quad v_{ii} = r^2 e^{rx^1}\delta_{1i}, \quad \textit{and} \quad v_{ij} = 0, \forall i \neq j.$$

Thus

$$Lv = a^{11}r^2 e^{rx^1} + b^1 r e^{rx^1} = (a^{11}r^2 + b^1 r)e^{rx^1}.$$

Since $a^{11} > 0$, we can choose $r > 0$ large enough such that $Lv > 0$, then for any $\epsilon > 0$, we have

$$L(u + \epsilon v) = Lu + \epsilon Lv > 0.$$

So by the result of the simple case, we get

$$\sup_{\Omega}(u+\epsilon v) = \sup_{\partial\Omega}(u+\epsilon v).$$

Now we let $\epsilon > 0$, we get

$$\sup_{\Omega} u = \sup_{\partial\Omega} u.$$

For the second part, the proof is just the same. ■

To generalize the theorem, we define

$$u^+ = \max\{u, 0\}, \quad u^- = u - u^+, \quad \Omega^+ = \{x | u(x) > 0\}.$$

Theorem 2 *With the same assumption as above, and suppose $Lu \geq 0$, $c \leq 0$, then*

$$\sup_{\Omega} u \leq \sup_{\partial\Omega} u^+.$$

If $Lu \leq 0, c(x) \leq 0$ instead, then

$$\inf_{\Omega} u \geq \inf_{\partial\Omega} u^-.$$

In particular, if $Lu = 0, c(x) \leq 0$, then

$$\sup_{\Omega} |u| = \sup_{\partial\Omega} |u|.$$

Proof: Let $L_0 u = a^{ij} D_{ij} u + b^i D_i u$, then in Ω^+ we have $L_0 u \geq -c(x)u \geq 0$. Thus by the previous theorem, we have

$$\sup_{\Omega^+} u = \sup_{\partial\Omega^+} u.$$

So

$$\sup_{\Omega} u = \sup_{\Omega} u^+ = \sup_{\Omega^+} u^+ = \sup_{\Omega^+} u = \sup_{\partial\Omega^+} u \leq \sup_{\partial\Omega} u^+. \quad ■$$

Uniqueness of solutions to Dirichlet problem

Corollary 1 *(**Uniqueness**) Suppose L elliptic, $c(x) \leq 0$, $u, v \in C^2(\Omega) \cap C^0(\overline{\Omega})$, and*

$$\begin{cases} Lu = Lv & , \quad in \quad \Omega, \\ u = v & , \quad on \quad \partial\Omega, \end{cases}$$

then $u = v$ in Ω.

*(**Comparison theorem**) If*

$$\begin{cases} Lu \geq Lv & , \quad in \quad \Omega, \\ u \leq v & , \quad on \quad \partial\Omega, \end{cases}$$

then $u \leq v$ in Ω.

A Priori C^0 estimates for solutions to $Lu = f$, $c \leq 0$.

Theorem 3 *Suppose L is strictly elliptic, $c(x) \leq 0$, $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$, where Ω is bounded domain.*
If $Lu \geq f$, then there exists constant $C = C(\lambda, \Omega)$ s.t.

$$\sup_{\Omega} u \leq \sup_{\partial\Omega} u^+ + C \sup_{\Omega} |f^-|.$$

If $Lu = f$, then

$$\sup_{\Omega} |u| \leq \sup_{\partial\Omega} |u| + C \sup_{\Omega} |f|.$$

Proof: Let $L_0 = a^{ij} D_{ij} + b^i D_i$, then

$$L_0 e^{rx^1} = (a^{11} r^2 + b^1 r) > \delta > 1$$

for r large enough. Let

$$v = \sup_{\partial\Omega} u^+ + (e^{rd} - e^{rx^1}) \sup_{\Omega} |f^-|,$$

where $d > x^1$ for $\forall x \in \Omega$. Then

$$Lv = L_0 v + cv \leq L_0 v \leq -\delta \sup_{\Omega} |f^-| \leq -\sup_{\Omega} |f^-|.$$

$$\therefore \quad L(v - u) \leq -\sup_{\Omega} |f^-| - f \leq 0, \quad in \quad \Omega.$$

But $v \geq u$ on $\partial\Omega$ by definition. Thus the last corollary tells us $v \geq u$ in Ω, i.e.

$$\sup_{\Omega} u \leq \sup_{\partial\Omega} u^+ + C \sup_{\Omega} |f^-|.$$

If $Lu = f$, replacing u by $-u$ and f by $-f$, we thus get the second result. ■

Strong maximum principle

First we introduce the Hopf's lemma.

Lemma 1 *Suppose L is uniformly elliptic, $c = 0$, $Lu \geq 0$ in Ω.*
Let $x_0 \in \partial\Omega$ be such that (i) u is continuous at x_0;
(ii) $u(x_0) > u(x), \quad \forall x \in \Omega$;
(iii) $\partial\Omega$ satisfies an interior sphere condition at x_0.
Then the outer normal derivative of u at x_0, if exists, satisfies

$$\frac{\partial u}{\partial \nu}(x_0) > 0.$$

If $c(x) \leq 0$, then it holds for $u(x_0) \geq 0$.
If $u(x_0) = 0$, then it holds for any $c(x)$.

Proof: Let $B(y,R)$ be the interior sphere, i.e. $B(y,R) \subset \Omega$ and $x_0 \in \partial B(y,R)$. Define $v(x) = e^{-\alpha r^2} - e^{-\alpha R^2}$, where $r = |x-y|$. Then

$$\begin{aligned}
Lv &= a^{ij} D_{ij} v + b^i(-\alpha(x^i - y^i)e^{-\alpha r^2}) \\
&= a^{ij}(-\alpha\delta^{ij} e^{-\alpha r^2} + \alpha^2 (x^i - y^i)e^{-\alpha r^2}) + b^i(-\alpha(x^i - y^i e^{-\alpha r^2})) \\
&= e^{-\alpha r^2}(\alpha^2 a^{ij}(x^i - y^i)(x^j - y^j) - \alpha a^{ii} - \alpha b^i (x^i - y^i)) \\
&> e^{-\alpha r^2}(\alpha^2 \lambda_0 r^2 - \alpha\Lambda - \alpha \sup|b| \cdot r)
\end{aligned}$$

Take $A = B_R(y) \setminus B_\rho(y)$, $0 < \rho < R$, then for α large enough, $Lv > 0$ in A.

The assumption (ii) tells us $u(x) < u(x_0)$ in Ω, in particular this holds on $\partial B(y,\rho)$, so there is some $\delta > 0$ s.t. $u(x) - u(x_0) < -\delta < 0$ on $\partial B_\rho(y)$.

Choose $\epsilon > 0$ s.t. $u(x) - u(x_0) + \epsilon v \leq 0$ on $\partial B_\rho(y)$.

Since $v = 0$ on $\partial B_R(y)$, we automatically have $u(x) - u(x_0) + \epsilon v \leq 0$ on $\partial B_R(y)$.

Also we have known

$$L(u - u(x_0) + \epsilon v) = Lu + \epsilon Lv > 0,$$

thus by the comparison theorem, we get

$$u - u(x_0) + \epsilon v \leq 0, \quad in \quad A.$$

So

$$\frac{\partial u}{\partial \nu}(x_0) \geq -\epsilon \frac{\partial v}{\partial \nu}(x_0) = \epsilon v'(R) > 0.$$

For $u(x_0) = 0$, just look at $L - c(x)$. ∎

Now we give the Strong Maximum Principle.

Theorem 4 *Suppose L is uniformly elliptic, $c = 0$, $Lu \geq 0$ in Ω, $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$. If u achieves its maximum in the interior, then u is constant.*

If $Lu \leq 0$ and u achieves its minimum in the interior, then u is constant.

If $c \leq 0$, then u cannot achieve a non-negative maximum in the interior unless u is constant.

Proof: Assume u is not constant, and achieves maximum M at x_0 in the interior.

Let $\Omega^- = \{x \in \Omega | u(x) < M\}$. By definition we know $\Omega^- \subset \Omega$, and $\partial\Omega^- \cap \Omega \neq \varnothing$ since u is not constant.

Let $x_1 \in \Omega^-$ be s.t. x_1 is closer to $\partial\Omega^-$ than $\partial\Omega$, and $B(x_1, R)$ be the largest ball in Ω^- centered at x_1. Then $u(y) = M$ for some $y \in \partial B(x_1, R)$.

By Hopf's lemma, we get

$$\frac{\partial u}{\partial \nu}(y) > 0.$$

This is a contradiction, since y is a maximum of u and so $Du(y)$ should be 0. ∎