Some Possible Questions to Explore

- Contagious structures for projections. In class we used Plunnecke inequality and Ruzsa inequality to prove contagious structure for projections of $A \times A \subset \mathbb{F}_p^2$. Are there similar results for projections of an arbitrary set $X \subset \mathbb{F}_q^2$. Here is a precise question. Suppose that $|\pi_t(X)| \le K|X|^{1/2}$ for $t = 0, \infty, t_1$, and t_2. Does it follow that $|\pi_{t_1+t_2}(X)| \le K^C |X|^{1/2}$ for a universal constant C? (What C can you get?) Similarly for $|\pi_{t_1 t_2}(X)|$ and $|\pi_{-t}(X)|$. See Lecture 11. (Possible reference: Katz-Tao work on "sums differences")
- Projections in algebraically independent directions. Suppose that $D = 0, 1, \infty, t_1, ..., t_r \subset \mathbb{R}$. Let $\pi_t(x_1, x_2) = x_1 + tx_2$. Let X be a finite subset of $\mathbb{R}^2$. Define

  $$S_D(N) = \min_{|X|=N} \max_{t \in D} |\pi_t(X)|.$$

  If $t_1, ..., t_r$ are algebraically independent over $\mathbb{Q}$, what upper and lower bounds can you prove on $S_D(N)$ (in terms of N and r)? Might want to start with $r = 1$.
- Optional question from pset 5, related to Bombieri-Vinogradov. In pset 5, using the large sieve, we proved the following estimate. If $X \subset [N]$, then for 90 % of $p \in P_{N^{1/2}}$,

  $$\|(\pi_p 1_X)_h^{*2}\|_{L^\infty(\mathbb{Z}_p)} \lessapprox |X|. \tag{1}$$

  This bound is sharp when X is an arithmetic progression of length N^α with $\alpha < 1/2$. But in this case, $\|1_X^{*2}\|_{\ell^\infty}$ is itself large. Suppose that $X \subset [N]$ with $|X| \sim N^{1/2}$, and suppose that $\|1_X^{*2}\|_{L^\infty} \lessapprox 1$. For most $p \in P_{N^{1/2}}$, can we prove a bound for $\|(\pi_p 1_X)_h^{*2}\|_{L^\infty(\mathbb{Z}_p)}$ which improves on (1)?
- Optional question from pset 4, related to the large sieve. To pursue this direction, it would be helpful to have a little background in restriction theory in Fourier analysis. In class, we used the large sieve to prove the following estimate.

  Theorem 1. *If $X \subset [N]$ and $|\pi_p(X)| \le (0.99)p$ for every $p \in P_{N^{1/2}}$, then $|X| \lessapprox N^{1/2}$*

  This theorem is essentially sharp when X is the set of squares. We could explore what happens if we know $|\pi_p(X) \le (0.99)p$ for every $p \in P_{N^\alpha}$ for some other exponent α, such as $\alpha = 1/4$. Or we could explore what happens if we replace $|\pi_p(X)| \le (0.99)p$ by a stronger bound like $|\pi_p(X)| \le N^{1/4}$ for every $p \in P_{N^{1/2}}$.
- Non-commutative projection theory. We have presented projection theory in the context of commutative groups. The setting is that we have a commutative group G and many homomorphisms $\pi_j : G \to H_j$. Each homomorphism can be described by its kernel, K_j.

  So $\pi_j : G \to G/K_j$. Now suppose that G is a non-commutative group. Let K_j be a bunch of subgroups, and consider the maps $\pi_j G \to G/K_j$. How much of what we discussed in class can be generalized to this setting? Might help to think in general or might help to pick a simple non-commutative group, such as $SL_2(\mathbb{F}_p)$. Projection theory for general commutative groups G is also a possible project to explore.
- Something else that you think of.