1. Introduction and overview

Tuesday Feb 4

In 18.156 this spring, we will study projection theory. Projection theory studies how a set X behaves under different orthogonal projections. Questions of this type aren't usually emphaisized in the graduate analysis curriculum, but they come up in many areas of math, including harmonic analysis, analytic number theory, additive combinatorics, and homogeneous dynamics. It is an especially good time to study projection theory, because there have been some striking recent applications, and because one of the central problems of the field was very recently solved. At the same time, there are many interesting open problems which I am excited to discuss and reflect on.

The goals of the course are:

- Learn the classical techniques and results of projection theory (with full details).
- Learn about applications in several areas.
- Learn about open questions.
- Learn some of the main ideas in the recent work in the field. Level of detail will depend on everyone's interest.

1.1. What is projection theory?

Suppose that we have a set $X \subset \mathbb{R}^n$. For any subspace $V \subset \mathbb{R}^n$, let $\pi_V : \mathbb{R}^n \to V$ denote the orthogonal projection. Projection theory studies the relationship between the properties of the set X and the properties of the projections $\pi_V(X)$ as V varies among k-dimensional subspaces. Informally, we are looking at X from many different points of view and trying to coordinate the different information.

The most basic question concerns the cardinality of X and the cardinality of $\pi_V(X)$ for different sets V. Suppose that X is a finite subset of $\mathbb{R}^2$, and write $|X|$ for the cardinality of a finite set. For almost every line L, $|\pi_L(X)| = |X|$, but there could be some special lines L where $|\pi_L(X)| < |X|$. For any number $S < |X|$, let $E_S(X)$ be the set of lines L with $|\pi_L(X)| \leq S$. The first question of projection theory is:

Question 1. *Suppose $X \subset \mathbb{R}^2$ is a finite set and $S < |X|$. Given $|X|$ and S, what is the maximum possible size of $E_S(X)$?*

A key example, suggested by Erdős, is when X is an integer grid. In this case, when the slope of L is a rational number of small height, $|\pi_L(X)|$ is small. Erdos conjectured that this example is the worst possible up to a constant factor, and in the early 1980s, Szemeredi and Trotter proved this conjecture.

Theorem 1.1. *(Szemeredi-Trotter 1982) If X is a finite subset of $\mathbb{R}^2$, and $S < \frac{1}{2}|X|$, then*

$$|E_S(X)| \leq CS^2|X|^{-1} + 1.$$

The proof of the Szemeredi-Trotter theorem uses topology, and it started an interesting interaction between combinatorial geometry questions and topology.

There are many variations of this question. For finite sets X, we can consider higher dimensions $\mathbb{R}^n$. Or we can consider other fields, like $X \subset \mathbb{F}_q^n$ where $\mathbb{F}_q$ is a finite field with q elements. Many of these questions are open.

We can also consider infinite sets X. This angle was taken in geometric measure theory, where the size of an infinite set is measured using Hausdorff dimension. We write $\mathrm{HD}(X)$ for the Hausdorff dimension of X. The question was first considered by Marstrand in the 1950s. He proved the following theorem.

Theorem 1.2. *(Marstrand, 1954) Is $X \subset \mathbb{R}^2$ is a compact set, then for almost every line L,*

$$HD(\pi_L(X)) = \min(HD(X), 1).$$

The lines L where $HD(\pi_L(X)) < \min(HD(X), 1)$ are called exceptional directions. Our second main question is to estimate the size of the set of exceptional directions. We let $E_s(X)$ be the set of lines L where $\mathrm{HD}(\pi_L(X)) < s$.

Question 2. *Suppose $X \subset \mathbb{R}^2$ and $s < \mathrm{HD}(X)$. Given $\mathrm{HD}(X)$ and s, what is the maximum possible Hausdorff dimension of $E_s(X)$?*

This second main question is called the exceptional set problem (for Hausdorff dimension). It is a geometric measure theory analogue of the first main question above, where size is measured by Hausdorff dimension instead of cardinality. In the 60s and 70s Kaufman and Falconer studied this question. Kaufman proved some results using a double counting argument, greatly simplifying the proof of Marstrand's theorem. And Kaufman and Falconer proved other results using Fourier analysis. These are the first fundamental results in the field. They are interesting and useful, but they don't give the full answer to Question 2. Nevertheless, no one improved on these results for about twenty years.

Furstenberg introduced a generalization of the exceptional set problem, which is called the Furstenberg set conjecture. Furstenberg was motivated by a question related to ergodic theory. Later Tom Wolff studied the exceptoinal set problem and the Furstenberg set conjecture. Wolff was motivated by the Kakeya conjecture and by other problems in geometric measure theory. Wolff studied the proof of Theorem 1.1 and tried to adapt the topological methods there to Question 2. He was able to prove some interesting estimates and he even applied them to prove some new estimates for the wave equation. But he was not able to prove any new estimates for

Question 2 itself. Wolff identified a key obstacle to addressing the exceptional set problem: the answer is different over $\mathbb{C}^2$ compared to $\mathbb{R}^2$, but most methods do not distinguish these two problems. Similarly, the projection problem in $\mathbb{F}_q^2$ is different depending on whether q is prime or not prime.

Around 2000, Bourgain proved the first estimates in projection theory that distinguish between $\mathbb{R}^2$ and $\mathbb{C}^2$. However, Bourgain's proof improves the previous exponents only by a tiny number ϵ. For the next twenty years, the bounds in the exceptional set problem were only tiny improvments of the old bounds of Kaufman and Falconer. But very recently, Question 2 was answered completely by Orponen, Shmerkin, Ren, and Wang.

Theorem 1.3. *(Orponen-Shmerkin-Ren-Wang) If $X \subset \mathbb{R}^2$, and $s < \mathrm{HD}(X)$, then*

$$\mathrm{HD}(E_s(X)) \leq \max(2s - \mathrm{HD}(X), 0).$$

The bound here is the natural analogue of the Szemeredi-Trotter theorem in the setting of Hausdorff dimension. There are many variations on this question too, and many of them are open. The field is developing rapidly.

1.2. **Applications of projection theory.** We will survey several applications of projection theory. For each topic, we will introduce and motivate the topic and see how it connects with projection theory. We will prove something about each topic but not necessarily the strongest results.

Sieve theory. Projection theory is closely parallel to some topics in sieve theory. Suppose now that $X \subset \mathbb{Z}$. For any integer q, let $\pi_q : \mathbb{Z} \to \mathbb{Z}/q\mathbb{Z}$ be the quotient map, which takes an integer n and outputs n mod q. Sieve theory studies the relationship between the properties of the set X and properties of $\pi_q(X)$ for different q.

Here is a sample result in sieve theory. One interesting example in sieve theory is the set of square numbers, which we denote as S. For every prime p, $|\pi_p(S)| = \frac{p+1}{2} \approx \frac{p}{2}$. Linnik proved that if $X \subset \{1, ..., N\}$ and $|\pi_p(X)| \leq \frac{p+1}{2}$ for every prime p, then $|X| \lessapprox N^{1/2}$. The set of square numbers up to N shows that Linnik's theorem is tight. The only known tight examples are close cousins of the square numbers, and it is an important open problem to understand whether there are other examples.

Another important direction in sieve theory is to understand how prime numbers are distributed modulo q for different q. Let P_x denote the set of prime numbers up to x. Dirichlet proved in the early 1800s that if q is fixed and $x \to \infty$, then P_x is evenly distributed modulo q among the residue classes that are relatively prime to q. Dirichlet's method only works when q is far smaller than x – the exact statement is messy but q needs to be smaller than x^ϵ for any $\epsilon > 0$. On the other hand, it is conjectured that for every $q \leq x^{1-\epsilon}$, the prime numbers are evenly distributed modulo q. The generalized Riemann hypothesis would imply that the prime numbers are evenly distributed modulo q for every $q \le x^{1/2-\epsilon}$.

Sieve theory leads to equidistribution results that hold for most q. In particular, Bombieri-Vinogradov proved that for almost all $q \le x^{1/2-\epsilon}$, the primes are evenly distributed modulo q. The point of sieve theory here is that we consider $\pi_q(P_x)$ for many different q and how these different "projections" are related to each other.

One important problem in this area is to try to understand the distribution of P_x mod q for most q when $q > x^{1/2}$. Yitang Zhang proved the first results of this kind in his proof of bounded gaps between primes. We will introduce this problem and some of the issues that make it difficult.

There is a close analogy between classical methods in projection theory and classical methods in sieve theory. Orthogonal projections $\pi_V : \mathbb{R}^n \to V$ and reduction modulo q, $\pi_q : \mathbb{Z} \to \mathbb{Z}/q\mathbb{Z}$ are both homomorphisms of Abelian groups. Much of projection theory only really depends on this homomorphism structure and so there are closely parallel results in the two settings. In particular, Falconer's work in projection theory (based on Fourier analysis) is closely analagous to the 'large sieve' method developed by Linnik and used by Bombieri-Vinogradov. And Kaufman's work in projection theory (based on double counting) is closely analogous to the 'larger sieve' method developed by Gallagher.

Sum-product problems. Suppose that A is a finite set of a field $\mathbb{F}$, such as $\mathbb{R}$ or $\mathbb{F}_p$. We write $A + A$ for the set of sums $\{a_1 + a_2 : a_1, a_2 \in A\}$ and we write $A \cdot A$ for the set of products $\{a_1 a_2 : a_1, a_2 \in A\}$. Erdos raised the question whether $\max(|A + A|, |A \cdot A|)$ must be much bigger than $|A|$. He conjectured that for any set $A \subset \mathbb{R}$, $\max(|A + A|, |A \cdot A|) \gtrsim |A|^{2-\epsilon}$, and Erdos and Szemeredi proved that there is some $c > 0$ so that $\max(|A+A|, |A \cdot A|) \gtrsim |A|^{1+c}$. Elekes connected the sum product problem to the Szemeredi-Trotter theorem and used the latter to prove a bound with a much better exponent: $\max(|A + A|, |A \cdot A|) \gtrsim |A|^{5/4}$.

Ever since Elekes's work, there has been a close connection between sum product problems and projection theory. This connection has been a two way street. Initially, Elekes used ideas from projection theory to prove new bounds in sum product theory. But the work of Bourgain and the recent work of Orponen-Shmerkin-Ren-Wang goes in the other direction, proving results in sum product theory first and then applying the results to projection theory in general.

Bourgain and Gamburd went on to apply these ideas in sum product theory to questions about random walks on finite groups such as $SL_2(\mathbb{F}_p)$. Suppose that $g_1, ..., g_k$ are a set of generators of $SL_2(\mathbb{F}_p)$ where we imagine that $k = O(1)$ and p is large. This set of generators determines a random walk on the group $SL_2(\mathbb{F}_p)$. Bourgain and Gamburd showed that, under fairly mild conditions on the generators, this random walk mixes very fast.

Homogeneous dynamics. The setting of homogeneous dynamics is a homogenous space such as $SL_n(\mathbb{R})/SL_n(\mathbb{Z})$. This homogeneous space can be viewed as the space of lattices in $\mathbb{R}^n$. It comes up in many problems in number theory. If $H \subset SL_n(\mathbb{R})$ is a Lie subgroup, and $x \in SL_n(\mathbb{R})/SL_n(\mathbb{Z})$, then we can consider the orbit $Hx \subset SL_n(\mathbb{R})/SL_n(\mathbb{Z})$, and we can ask how this orbit is distributed. If H is a unipotent subgroup, then there is a very rigid classification theorem due to Ratner, building on special cases proven by Dani and Margulis. Ratner's theorem says that either the orbit Hx is dense and evenly distributed, or else there is a very specific algebraic structure that describes the orbit.

Recently, Lindenstrauss and Mohammadi returned to this question and worked on proving good quantitative bounds in Ratner's theorem. So far, they were able to do so in some special cases. One of their key new ideas is to connect these problems in homogeneous dynamics with projection theory.

We will introduce this area, motivate the question, and learn how projection theory enters the story.

Those are all the applications that we had time to discuss in the class, but in this introduction, we briefly mention a couple of others.

Imaging. Projection theory also comes up in different imaging technologies, from CAT scans to Cryo-electron-microscopy. In these settings, one tries to reconstruct a set X or function f from some information about its projections. Some of the math involved involved in imaging technology is related to the math in this course. In particular, imaging technology makes use of the close connection between projection theory and Fourier analysis.

Fourier analysis. Projection theory has a close connection with Fourier analysis. Philosophically, projection theory is closely related to additive structure: the key feature of a projection $\pi_V : \mathbb{R}^n \to V$ is that it is a group homomorphism of abelian groups. Fourier analysis is also closely related to the additive structure of $\mathbb{R}^n$: in Fourier analysis we study the characters of an abelian group. This leads to nice formulas relating projections and Fourier transforms. We will use Fourier analysis in our study of projection theory.

Recent work in Fourier analysis, especially related to decoupling theory, is closely related to projection theory, and ideas have gone in both directions.

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