3. Projection theory for balls in Euclidean space

Tuesday February 11

In this lecture, we develop the tools from the last lecture in the more geometric setting of Euclidean space.

We first introduced our main tools in the setting of finite fields, where the technical details are simple. Now we adapt these tools to Euclidean space. Euclidean space has many different scales. We have to take into account many different scales in order to even ask good questions in Euclidean space. Paying attention to multiple scales will go on to be one of the key ideas in the subject.

We suppose that X is a set of disjoint balls in Euclidean space, and study the orthogonal projections of X in different directions. Here is the precise setup.

SETUP

Let X be a set of disjoint unit balls in $B_R \subseteq \mathbb{R}^2$. Let $D \subset S^1$ be a set of $1/R$ separated directions.

$$S = S(X, D) = \max_{\theta \in D} |\pi_\theta(X)|.$$

$$N_X(r) = \max_{c \in \mathbb{R}^2} |X \cap B(c, r)|.$$

$$N_D(\rho) = \max_{\substack{\sigma \in S' \text{arc} \\ |\sigma| = \rho}} |D \cap \sigma|.$$

Double Counting

Theorem 3.1. *(Double Counting Real Version)*
If ***SETUP****, then*

$$|D| \lesssim \frac{S}{|X|} \sum_{1 \le r \le R} N_X(r) N_D(1/r).$$

Proof.

$$* = \#\{B_1, B_2 \text{ unit balls } \in X, \theta \in D : \pi_\theta(B_1) \cap \pi_\theta(B_2) \neq \emptyset\}.$$

<u>Lower bound</u>: $* \gtrsim |D| \left(\frac{|X|}{S}\right)^2 S$. It basically follows from the same argument as in the finite field setting.

<u>Upper bound</u>: Fix B_1, B_2 with $dist(B_1, B_2) \sim r$, let $c(B_1)$, $c(B_2)$ be the center of B_1 and B_2. Write

$$v = \frac{c(B_2) - c(B_1)}{|c(B_2) - c(B_1)|}$$

to be the angle from B_1 to B_2. (see Figure 1) If $\pi_\theta(B_1) \cap \pi_\theta(B_2) \neq \emptyset$, then $angle(\theta, \gamma) \lesssim 1/r$. Thus,

$$\#\{\theta : \pi_\theta(B_1) \cap \pi_\theta(B_2)\} \lesssim N_D(1/r).$$

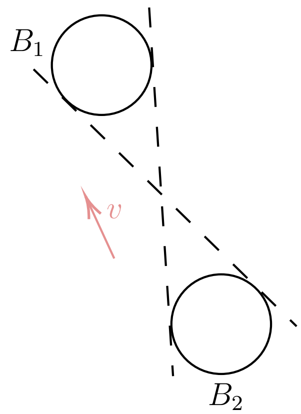


FIGURE 1. Angle between two balls

$$\#\{B_1, B_2 \in X : dist(B_1, B_2) \lesssim r\} \lesssim |X| N_X(r).$$

Thus,

$$* \lesssim \sum_{\substack{r \text{ dyadic} \\ 1 \le r \le R}} |X| N_X(r) N_D(1/r)$$

so

$$|X|^2 S^{-1} D \lesssim * \lesssim \sum_{\substack{r \text{ dyadic} \\ 1 \le r \le R}} |X| N_X(r) N_D(1/r).$$

□

Example 3.2. *For $N_X(r)$,*

(1) *X neighborhood of a curve. (see Figure 2a)*

$$N_X(r) \sim r$$

(2) *Well-spaced $N \times B$ grid. (see Figure 2b)*

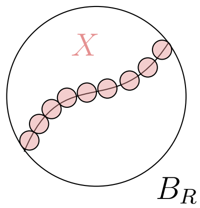
$$N_X(r) = \begin{cases} 1 & r \le R/N \\ r^2 \frac{N^2}{R^2} & r > R/N \end{cases}$$

(3) *A cluster of N^2 unit balls (see Figure 2c)*

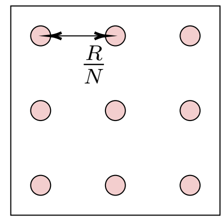
$$N_X(r) \sim \begin{cases} r^2 & r \le N \\ N^2 & r > N \end{cases}$$

Pictures of How $N_r(X)$ depends on r (see Figure 3)

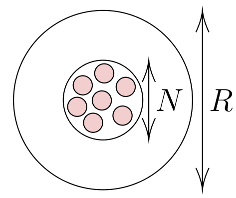
Normalize $N = R^{1/2}$.



(A) A neighborhood of a curve.



(B) A well-spaced grid.



(C) A cluster of balls.

FIGURE 2. Examples for $N_X(r)$.

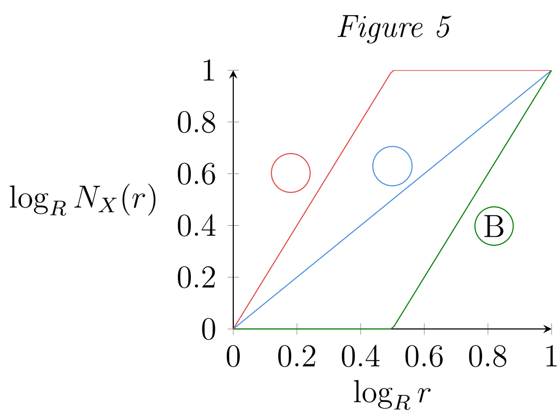


FIGURE 3. Plots of $N_r(X)$ vs r.

Straight Line Case

$$|X| = R^\alpha, N_X(r) \sim r^\alpha$$

We call this regular α dim spacing.
Below Straight Line Case

$$|X| = R^\alpha, N_X(r) \lesssim r^\alpha$$

We call this α dim spacing.

Definition 3.3. *We say that X has Hausdorff spacing if it has α dimension spacing for some α. Another way to say this is that*

$$N_{R^\beta}(X) \lesssim |X|^\beta$$

for any $0 \leq \beta \leq 1$.

Corollary 3.4. *(Double Counting Real Version)*
*If **SETUP** X, D has Hausdorff spacing then*

$$|D| \lesssim \log R(|S| + \frac{|S|}{|X|}|D|) \Rightarrow (S \sim X \text{ or } |D| \lesssim S).$$

Proof. Let's calculate $N_X(r)N_D(1/r)$. Suppose $r = R^\beta$, the Hausdorff condition implies

$$N_X(R^\beta)N_D(R^{-\beta}) \lesssim |X|^\beta |D|^{1-\beta}$$

Thus, by theorem 3.1,

$$|D| \lesssim \log R \frac{|S|}{|X|} \max_{0\leq\beta\leq 1} |X|^\beta |D|^{1-\beta} \lesssim \log R(|S| + \frac{|S|}{|X|})|D|.$$

□

Recall the theorem in the finite field case.

Theorem 3.5. *If $X \subseteq \mathbb{F}_q^2$, $D \subseteq \mathbb{F}_q, S = \max_{\theta\in D} |\pi_\theta(X)|$ then $S \sim |X|$ or $|D| \lesssim S$.*

Note that in the $\mathbb{R}$ setting if we impose the Hausdorff spacing condition, then we get basically the same result as in the finite field case.
Now let's compare result in projection theory in $\mathbb{F}_q^2$ vs unit balls in B_R^2 with Hausdorff spacing.

Theorem 3.6. *(Fourier Method Finite Field) If $\mathbb{F}_q$-SETUP and $S \leq q/2$, then*

$$|D| \lesssim \frac{Sq}{|X|}.$$

Corollary 3.7. *If $\mathbb{R}$-SETUP and X, D has Hausdorff spacing. Then, $|D| \lesssim \frac{SR}{|X|}$*

Conjecture 3.8. *If p primes, $\mathbb{F}_p$ SETUP and $S \leq \frac{1}{2}\min(q, |X|)$ then*

$$|D| \lesssim \frac{|S|^2}{X}$$

Conjecture 3.9. *(Furstenberg) If SETUP, X and D has Hausdorff spacing and*

$$S \leq R^{-\epsilon}\min(R, |x|),$$

then

$$|D| \lessapprox \frac{|S|^2}{|R|}.$$

The above conjecture is proven in 2024 by (Orponen, Shmerkin, Ren and Wang)

3.1. **Fourier Method.**

Lemma 3.10. *(Main lemma in finite field)*
If $\mathbb{L}$ *is a set of lines in* $\mathbb{F}_q^2$. *Write* $f = \sum_{L\in\mathbb{L}} 1_L(x)$. *Then,* $f = f_0 + f_1$ *so* $supp\hat{f}_0 = \{0\}$, $supp\hat{f}_h = \{0\}^c$. *Then,* f_0 *is a constant function. Then,* $\|f_0\|_2^2 = |\mathbb{L}|^2$, $\|f_h\|_2^2 = |\mathbb{L}|q$.

Now, let's look at the $\mathbb{R}$ setting. Let $\mathbb{T}$ be a set of $1 \times R$ in $\mathbb{R}^2$. Let ϕ_T be a smooth approximation of 1_T.

Lemma 3.11. *(Main lemma in real)*
Let $\mathbb{T}$ *be a set of* $1 \times R$ *rectangles in* $\mathbb{R}^2$. *Let* $f = \sum_{T\in\mathbb{T}} \phi_T(x)$. *Then,*

$$f = \sum_{\substack{1\le r\le R \\ dyadic}} f_r(x)$$

such that $supp\hat{f}_r \subseteq B(1/r)$ *and* $\|f_r\|_2^2 \lesssim N_{\mathbb{T}}(r)|\mathbb{T}|r^{-1}R$ *where*

$$N_{\mathbb{T}}(r) := \max_{\tilde{T}:2r\times 2R rect} \#\{T \in \mathbb{T} : T \subset \tilde{T}\}.$$

Proof. (proof sketch of main lemma)
$\text{supp}\hat{\phi}_T \subset T^*$ where $T^* := \{\xi \in \mathbb{R}^2 : |(x_1 - x_2)\cdot\xi| \le 1, \text{ any } x_1, x_2 \in T\}$. (see Figure 4) □

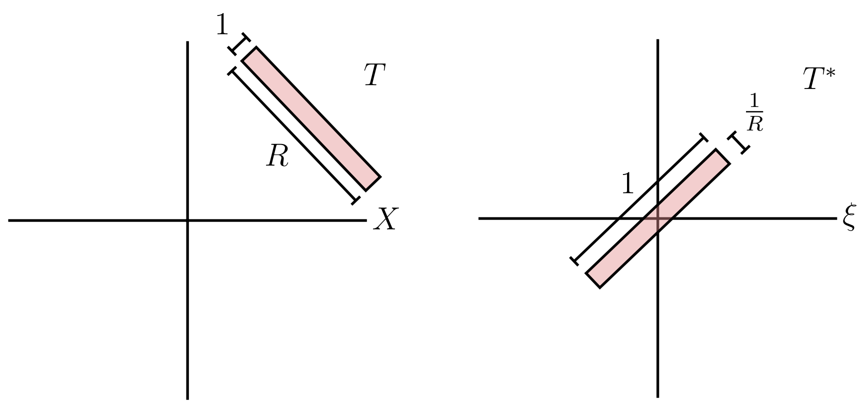


FIGURE 4. The dual of a rectangle.

Littlewood-Paley decomposition
Write $1 = \sum_{\substack{1\le r\le R \\ dyadic}} \eta_r(\xi)$ with $\eta_r \ge 0$ such that

$$\text{supp}\eta_r \subseteq Ann(\frac{1}{10r} \le |\xi| \le \frac{1}{r}), 1 < r < R$$

and supp$\eta_R \subseteq B(1/R)$ and supp$\eta_1 \subseteq \{\xi : |\xi| > 1/10\}$. Define $f_r = (\eta_r \hat{f})^\vee$ so supp$\hat{f}_r \subseteq B(1/r)$. In particular, we can write $\phi_{T,r} = (\eta_r \hat{\phi}_T)^\vee$.

Visual of $\hat{\eta_r}$ and $\phi_{T,r}$

We have $\eta_r(\xi) \sim 1$ on $Ann(1/r)$ and

$$|\check{\eta}_r(x)| \sim \begin{cases} 1/r^2 \text{ on } |x| \lesssim r \\ \text{rapidly decay if } |x| > r. \end{cases}$$

(see Figure 5) where $\check{\eta}_r(x) = \int e^{ix\cdot\xi}\eta_r(\xi)d\xi$. Note that $\int \check{\eta}_r(x)dx = \eta_r(0)$. As

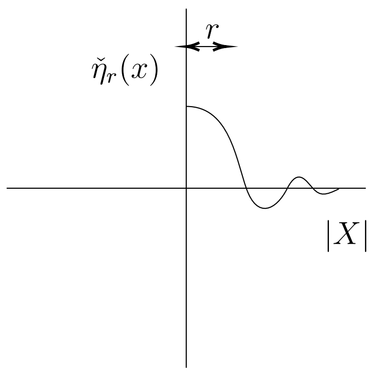


FIGURE 5. Visual of radial component of $\hat{\eta_r}$.

$\int |\check{\eta}_r(x)|dx = \eta_r(0)$, we have that $f * |\check{\eta}_r(x)| \sim$ Average of f on $B(x,r)$. As $\phi_{T,r} = \phi_T * \check{\eta}_1$, we have $|\phi_{T,r}(X)| \sim r^{-1}1_{r \text{ neighborhood of } T}$. (see Figure 6)

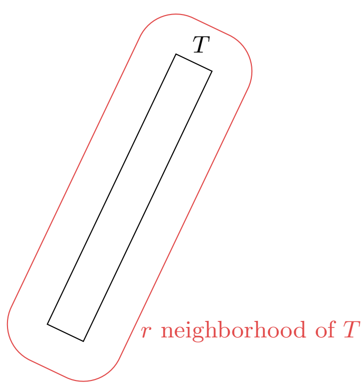


FIGURE 6. A tube T and its r neighborhood.

Lemma 3.12. *(Orthogonality)*
If T_1, T_2 are $1 \times R$ tubes then $|\langle \phi_{T_1,r}, \phi_{T_2,r} \rangle| \lesssim R^{-1000}$ unless there exists $\tilde{T}$, a $R^\epsilon r \times R^{1+\epsilon}$ rectangle such that $T_1, T_2 \in \tilde{T}$.

Proof. (proof sketch) If $angle(T_1, T_2) \gtrsim R^\epsilon \frac{r}{R}$, then $\text{supp}\hat{\phi_{T_1,R}} \cap \text{supp}\hat{\phi_{T_2,R}} = \emptyset$. If $N_r(T_1)$ and $N_r(T_2)$ are disjoint, then

$$\int \phi_{T_1,r}\phi_{T_2,r} = \int \phi_{T_1} * \hat{\eta}_r \phi_{T_1} * \hat{\eta}_r \lesssim R^{-1000}$$

as $\phi_{T_1} * \hat{\eta}_r$ and $\phi_{T_1} * \hat{\eta}_r$ have essentially disjoint support. □

$\mathbf{L^2}$ estimates

$$\|f_r\|_2^2 = \|\sum_{T\in\mathbb{T}} \phi_{T,r}\|_{L^2}^2 \tag{10}$$

$$= \sum_{T_1,T_2} \langle \phi_{T_1,r}, \phi_{T_2,r} \rangle \tag{11}$$

$$= \sum_{T_1 \sim_r T_2} \langle \phi_{T_1,r}, \phi_{T_2,r} \rangle + neglible \tag{12}$$

$$\leq \sum_{T_1 \sim_r T_2} \|\phi_{T,r}\|_2^2 + \|\phi_{T_2,r}\|_2^2 \tag{13}$$

$$\leq N_r(\mathbb{T}) \sum_{T\in\mathbb{T}} \|\phi_{T,r}\|_2^2 \tag{14}$$

$$= N_{\mathbb{T}}(r) \sum_{T\in\mathbb{T}} \|\phi_{T,r}\|_2^2 \tag{15}$$

$$= N_{\mathbb{T}}(r)|\mathbb{T}|r^{-2}rR \tag{16}$$

where r^{-2} is the amplitude and rR is the area.