4. The Fourier method in Euclidean space

Thursday February 13. Transcribed by Jacob Reznikov. Used with permission.

In this lecture, we finishing developing the Fourier method for projection estimates in Euclidean space.

Before we dive into the Fourier method in Euclidean space, let us overview the result in the case of finite fields. The main lemma used in the finite case is the following.

Lemma 4.1 (Main Lemma 2F)**.** *If $\mathbb{L}$ is a collection of lines in $\mathbb{F}_q^2$, and $L(x)$ is the characteristic function for $L \in \mathbb{L}$, then we can decompose*

$$f(x) = \sum_{L \in \mathbb{L}} L(x)$$

as $f = f_0 + f_h$, where $f_0 = \frac{|\mathbb{L}|}{q}$, f_0 is orthogonal to f_h, and

$$||f_0||_{L^2}^2 \lesssim |\mathbb{L}|^2 , \quad ||f_h||_{L^2}^2 \lesssim |\mathbb{L}|q \, .$$

Now one can use this lemma to get L^2 bounds on f quite easily, we immediately get $||f||_{L^2}^2 \lesssim ||f_0||_{L^2} + ||f_h||_{L^2}$, however, there are easier ways to get this same bound.

Lemma 4.2 (Elementary L^2 bounds on f)**.** *We have $||f||_{L^2}^2 \lesssim |\mathbb{L}|q + |\mathbb{L}|^2$.*

Proof. We can directly compute

$$\begin{aligned} ||f||_{L^2}^2 &= \sum_{x \in \mathbb{F}_q^2} \left[\sum_{L \in \mathbb{L}} L(x) \right]^2 = \sum_{x \in \mathbb{F}_q^2} \left[\sum_{L_1, L_2 \in \mathbb{L}} L_1(x) L_2(x) \right] \\ &= \sum_{x \in \mathbb{F}_q^2} \left(\left[\sum_{L_1 = L_2 \in \mathbb{L}} L_1(x) L_2(x) \right] + \left[\sum_{L_1 \neq L_2 \in \mathbb{L}} L_1(x) L_2(x) \right] \right) \end{aligned}$$

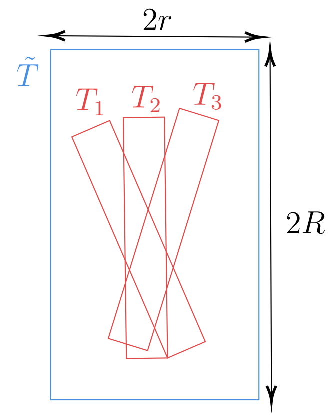
Now different lines always meet at exactly one point, so $\sum_{x \in \mathbb{F}_q^2} L_1(x) L_2(x) = 1$ for $L_1 \neq L_2$. Thus we have

$$||f||_{L^2}^2 \leq \sum_{x \in \mathbb{F}_q^2} \left(\left[\sum_{L \in \mathbb{L}} L^2(x) \right] \right) + \sum_{L_1 \neq L_2 \in \mathbb{L}} 1 \leq |\mathbb{L}|q + |\mathbb{L}|^2$$

□

One could then ask, isn't the Fourier method then useless if we can arrive at the same norm bound in an easier way? And in some regimes, it is, if $|\mathbb{L}| \sim q$ then the Main Lemma does not give us any extra information. However, in the case where $|\mathbb{L}| \gg q$ we not only get the L^2 bounds, but we also get the extra piece of information the constant part, the zeroth frequency, of f, dominates the contributions to the norm. We can interpret this information as asserting that f is in some sense 'almost constant'. The usefulness of this will become clear in the Euclidean case.

We now recall the setup for the Fourier Method in Euclidean Space.



Setup

Suppose that $\mathbb{T}$ is a set of $1 \times R$ rectangles.

Suppose that for each rectangle $T \in \mathbb{T}$, ψ_t is a smooth approximation for 1_T.

Let $f = \sum_{T \in \mathbb{T}} \psi_T$ and $N_{\mathbb{T}}(r) = \max_{\tilde{T}} |\{T \in \mathbb{T} : T \subset \tilde{T}\}|$ where $\tilde{T}$ ranges across all $2r \times 2R$ rectangles, as can be seen in the diagram on the right.

Lemma 4.3 (Main Lemma 2R). *If the setup holds then we can decompose f as*

$$f = \sum_{\substack{1 \le r \le R \\ r \text{ dyadic}}} f_r$$

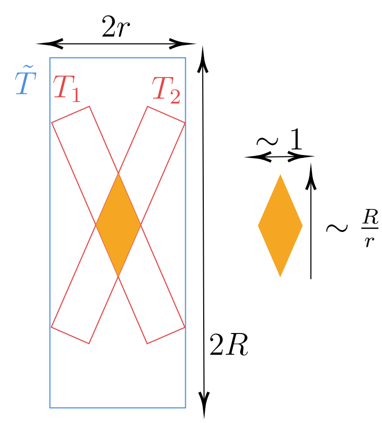
with f_r (nearly) orthogonal to each other, and for each r,

$$\hat{f}_r \subset B(1/r) \quad \text{and} \quad ||f_r||_{L^2}^2 \lesssim |\mathbb{T}| N_{\mathbb{T}}(r) \frac{R}{r}$$

Now again we can use this lemma to arrive at a quick L^2 bound, simply adding up over r we get $||f_r||_{L^2}^2 \lesssim \sum_{r \text{ dyadic}} |\mathbb{T}| N_{\mathbb{T}}(r) \frac{R}{r}$. But once again, there are easier ways to get this bound, which we will now show.

For two tubes T_1, T_2 we will write $r(T_1, T_2)$ to be the minimal r such that T_1 and T_2 are both contained in a $2r \times 2R$ rectangle.

A simple look at the geometry of the rectangles gives us the following lemma



Lemma 4.4. *For any two tubes T_1, T_2 we have*

$$\int T_1(x) T_2(x) dx \sim \frac{R}{r(T_1, T_2)}$$

In a similar way to the elementary bound in the finite case we can compute directly, we will use the previous lemma, and group the terms in the sum by r

$$\int f^2 = \sum_{T_1, T_2 \in T} \int T_1(x) T_2(x) dx = \sum_{r \text{ dyadic}} \sum_{\substack{T_1, T_2 \in T \\ r \sim r(T_1, T_2)}} \frac{R}{r}$$

Now fix r, for the first tube we have $|\mathbb{T}|$ choices and for the second we have at most $N_{\mathbb{T}}(r)$ choices. This gives us

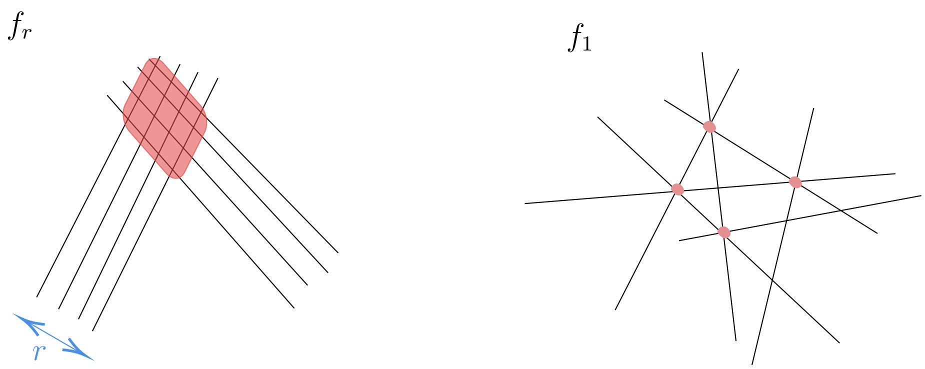
$$\sum_{r \text{ dyadic}} \sum_{\substack{T_1,T_2 \in T \\ r \sim r(T_1,T_2)}} \frac{R}{r} = \sum_{r \text{ dyadic}} |\mathbb{T}| N_{\mathbb{T}}(r) \frac{R}{r} .$$

Once again we get the same L^2 bound as from the Main Lemma.

Thus we again find that the important part of the Lemma, isn't just the L^2 bound, its the extra information we get about the frequency structure of the function. We will want to think about this information in a particular way, which we will call the 'locally constant intuition'.

Intuition If supp$\hat{g} \subset B_{1/r}$ then $g \approx$ constant on each B_r. This intuitively should make sense, if supp$\hat{g} \subset B_{1/r}$ then g is a combination of waves with frequency at most $1/r$, since each wave is then approximately constant on any given B_r then it is plausible that their combination is as well.

Now to use this intuition in our setup let us consider the following diagrams



The left diagram shows us what happens in a setup where our f is dominated by some f_r with r large, our function then is dominated by the scale r and we can see as expected by our intuition, that for most balls of radius r, our function is relatively constant. Furthermore, the locations where f is large will all look like the blob we have drawn in red, and they will have more geometric structure to exploit there.

On the other hand when f is dominated by f_1, it is dominated by high frequencies and it might look like the diagram on the right, here we have less points where f is large but they are more scattered and have less structure.

Now let us formalize this intuition before using it with our main lemma. Consider a function g with supp$\hat{g} \subset B_{1/r}$, what can we say about it? Well in analysis there is often a specific way we deal with supports we know, and that is using a bump function. That is, let η be a compactly supported smooth function with $\eta = 1$ on $B(1,r)$, then we have $\hat{g} = \hat{g} \cdot \eta$ and so applying inverse Fourier to this equation we get $g = g * \check{\eta}$. We will need three important properties of of $\check{\eta}$.

- $|\check{\eta}(x)| \lesssim r^{-2}$, which comes from simple triangle inequality applied to the integral defining $\check{\eta}$.
- $|\check{\eta}(x)| \lesssim r^{-2}(\frac{|x|}{r})^{-1000}$, which comes from integration by parts.
- If η is radial, then $\check{\eta}$ is also radial, which we will assume to be the case henceforth.

Now we can use these two facts to get information about g. We define $\psi_r := |\check{\eta}|$ and derive the following.

Lemma 4.5. *If* $supp \hat{g} \subset B_{1/r}$*, then* $|g(x)| \leq |g| * \psi_r$.

Proof. We compute

$$|g(x)| = |(g * \check{\eta})(x)| = \left| \int g(y)\check{\eta}(x-y) \right| \leq \int |g(y)||\check{\eta}(x-y)| = |g| * \psi_r$$

□

Lemma 4.6. *If* $supp \hat{g} \subset B_{1/r}$*, then* $|g(x)|^2 \lesssim |g|^2 * \psi_r$.

Proof. We again compute

$$|g(x)| = |(g * \check{\eta})(x)|^2 = \left| \int g(y)\check{\eta}(x-y) \right|^2$$

Now we write $g(y)\check{\eta}(x-y) = (g(y)\check{\eta}(x-y)^{1/2}) \cdot (\check{\eta}(x-y)^{1/2})$ and apply Cauchy Schwarz to get

$$\begin{aligned} \left| \int g(y)\check{\eta}(x-y) \right|^2 &\leq \int (g(y)\check{\eta}(x-y)^{1/2})^2 \int (\check{\eta}(x-y)^{1/2})^2 \\ &= \int g(y)^2 \check{\eta}(x-y) \int \check{\eta}(x-y) \\ &\lesssim (|g|^2 * \psi_r)(1) \end{aligned}$$

□

Back to our setup, we can now apply all these computations to improve our L^2 bound and derive the Euclidean version of theorem 2F. We recall our setup.

Setup. X is a set of unit balls in $B_R \subset \mathbb{R}^2$.

$D \subset S^1$ is a set of directions, which is $1/R$-separated.

$S = \max_{\theta \in D} |\pi_\theta(X)|$.

$N_X(r) = \max_{c \in \mathbb{R}^2} |X \cap B(c,r)|$ and $N_D(\rho) = \max_{\substack{\sigma \subset S^1 \\ |\sigma| = \rho}} |D \cap \sigma|$.

We will use $\lessapprox$ to mean $g(R,x) \leq C \log(R) f(R,x)$ for some constant C.

Theorem 4.7. *If our setup holds then*

$$|D| \lessapprox \frac{SR}{|X|} \max_r \frac{N_X(r)N_D(r/R)}{r^2}.$$

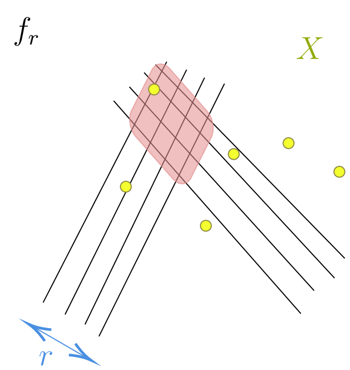
Proof. First for all $\theta \in D$ we define $\mathbb{T}_\theta$ to be the set of S different $1 \times R$ tubes T at angle θ that cover X. We then set

$$\mathbb{T} = \bigcup_{\theta \in D} \mathbb{T}_\theta \quad f(x) = \sum_{T \in \mathbb{T}} \psi_T(x).$$

Then for any $x \in X$ we have $|f(x)| \geq |D|$ so we get the simple lower bound

$$|D|^2|X| \leq \int_X |f|^2$$

Now the upper bound will be a bit trickier, let us think again about the picture we had before, and notice that if our X set is quite spread apart, that is when $N_{\mathbb{T}}(r)$ is small, then estimating $\int_X |f_r|^2$ by $||f_r||_{L^2}^2$ will be quite a lossy comparison, we can do better.



First we will use the fact that $\text{supp}\hat{f}_r \subset B_{1/r}$ to get

$$\int_X |f_r|^2 = \int 1_X |f_r|^2 dx \leq \int 1_X \cdot (|f_r|^2 * \psi_r) dx = \int \int 1_X(x)|f_r|^2(y)\psi_r(x-y)dydx$$

Now let us assume that η and hence ψ_r are radial, then they are also symmetric, so this entire expression is symmetric with respect to swapping x and y. Hence we have

$$\int_X |f_r|^2 = \int |f_r|^2(x) \int 1_X(y)\psi_r(x-y)dydx = \int |f_r|^2(x)(1_X * \psi_r)dx$$

Now morally ψ_r is approximately $r^{-2}1_{B_{1/r}}$ so we have that $1_X * \psi_r \lesssim r^{-2}N_X(r)$. This then gives us

$$\int_X |f_r|^2 \lesssim r^{-2}N_X(r) \int |f_r|^2 \lesssim \frac{R|\mathbb{T}|N_{\mathbb{T}}(r)N_X(r)}{r^3}.$$

Now let us estimate $N_{\mathbb{T}}(r)$, for any fixed θ we know that the number of rectangles of size $1 \times R$ that can fit inside a rectangle of size $2r \times 2R$ is $\lesssim r$ since no more can fit. The maximum angle (with respect to the large rectangle) that can fit is going to be $\lesssim r/R$, so as many as $N_D(r/R)$ different θ can count, hence we have a bound of $N_{\mathbb{T}}(r) \lesssim rN_D(r/R)$. We thus have

$$\int_X |f_r|^2 \lesssim \frac{R|\mathbb{T}|N_D(r/R)N_X(r)}{r^2}.$$

We also have $|\mathbb{T}| = S|D|$ so putting it all together we have

$$\begin{aligned} |X||D|^2 &\leq \sum_{\substack{1\leq r\leq R \\ r \text{ dyadic}}} \int_X |f_r|^2 \lesssim S|D|R \sum_{\substack{1\leq r\leq R \\ r \text{ dyadic}}} \frac{N_D(r/R)N_X(r)}{r^2} \\ &\leq S|D|R \log R \max_{\substack{1\leq r\leq R \\ r \text{ dyadic}}} \frac{N_D(r/R)N_X(r)}{r^2} \end{aligned}$$

which we can rewrite into

$$|D| \lessapprox \frac{SR}{|X|} \max_{\substack{1\leq r\leq R \\ r \text{ dyadic}}} \frac{N_D(r/R)N_X(r)}{r^2}$$

□

Now this result looks a little ugly, so let us see what it looks like with the Hausdorff assumption we discussed last class. Recall that we say X has Hausdorff spacing if $N_X(R^\beta) \lesssim |X|^\beta$ for all $0 \leq \beta \leq 1$. If then X and D both have Hausdorff spacing then we have

$$\max_{\substack{1\leq r\leq R \\ r \text{ dyadic}}} \frac{N_D(r/R)N_X(r)}{r^2} \sim 1 + \frac{|X||D|}{R^2}$$

Corollary 4.8. *If the setup holds and X, D both have Hausdorff spacing then*

$$|D| \lessapprox \frac{SR}{|X|} + \frac{S|D|}{|X|}.$$

In particular either $R \lessapprox S$ or $|D| \lessapprox \frac{SR}{|X|}$.

We will end off this section with a little bit of history about the Fourier and double counting method.

Fourier Method History

- 1940s - First use of Fourier method by Linnik in Sieve Theory.
- 1970s - Fourier method use by Rot for the Heilbronn triangle problem.
- 1980s - Falconer uses the method for geometric measure theory (what we are currently doing).

- Recently - Vinh used the Fourier method in the finite field setting.

Double Counting Method History

- 60s - Kaufmann uses double counting method for geometric measure theory.
- 60s - Gallagher uses double counting method for Sieve theory.

4.1. **Sieve Theory.** We will now move on to the study of Sieve theory, which as we will see is very similar to what we have done so far.

We will be interested in studying the maps $\pi_q : \mathbb{Z} \to \mathbb{Z}_q := \mathbb{Z}/q\mathbb{Z}$ given by $\pi_q(x) = x \mod q$. These will play the role of our projections, in the sense that they are also group homomorphisms of Abelian groups.

We will use $[N]$ to denote the set $\{1, ..., N\}$ and we will study the projections of subsets of $[N]$.

Example Consider the set $X = \{n^2 : 1 \leq n \leq N^{1/2}\} \subset [N]$, we know from basic algebra that $|\pi_p(X)| = \frac{p+1}{2}$ for all primes p. This should seem unusual since we could have an extremely large set and yet all of its projections miss half of their co-domain. The natural next question is, how large can a set S be and still have this property?

Theorem 4.9 (Linnik)**.** *If $X \subset [N]$, $\pi_p(X) \leq \frac{p+1}{2}$ for all prime p then $|X| \lessapprox N^{1/2}$.*

The only known sharp families for this theorem are square numbers and their close relatives, namely images of specific quadratic polynomials.

Let us now begin analyzing this problem using the double counting method.

Theorem 4.10 (1S)**.** *If $X \subset [N]$, D a set of primes less than N and for all $p \in D$ we have that $|\pi_p(X)| \leq S$, then either $|X| \leq 2S$ or $|D| \lessapprox S$.*

Proof. We start as usual by considering the set of coincidences

$$(*) = \{x_1, x_2 \in X, p \in D : \pi_p(x) = \pi_p(x_2)\}$$

by the same argument as usual we have the lower bound

$$|(*)| \geq |D| \left(\frac{|X|}{S}\right)^2 = |X|^2|D|S^{-1}.$$

For the upper bound fix x_1 and x_2 and count the number of p's for which the condition can hold, if $\pi_p(x_1) = \pi_p(x_2)$ then we have $p|x_2 - x_1$. We now have two cases

If $x_1 = x_2$ then any p works, this gives us a $|X||D|$ term.

If $x_1 \neq x_2$ then only the prime divisors of $x_1 - x_2$ work of which there are at most $\log N$, so this gives us an $|X|^2 \log N$ term.

Together we get

$$|X|^2|D|S^{-1} \leq |(*)| \leq |X||D| + |X|^2 \log N$$

which we can rewrite as

$$|D| \leq \frac{S|D|}{|X|} + S \log N$$

so either the first term dominates and we have $S \leq 2|X|$ or the second term dominates and we get $|D| \lessapprox S$. □

As an example if $|\pi_p(X)| \leq N^{2/3}$ for any $p \in D$ with $|D| = O(\log N)N^{2/3}$ then $|X| \lessapprox N^{2/3}$.