6. Projections and smoothing

Tues Feb 25. Transcribed by Hang Du. Used with permission.

The projection of a rough function at a typical angle is usually smoother than the original function. This fundamental observation is one of the core principles of projection theory. It is also closely related to the large sieve.

We first set up what it means to project a function and then state the result precisely. The proof is closely analogous to the proof of the large sieve.

Setup
Let $f : \mathbb{R}^d \to \mathbb{C}$ be a L^2 function. For $V \subset \mathbb{R}^d$ a subspace, we define

$$\pi_V f(y) = \int_{V^\perp} f(y+z)\,\mathrm{dvol}_{V^\perp}(z)\,, \forall y \in V\,.$$

In particular, for $\theta \in \mathbb{S}^{d-1}$ we write $\pi_\theta f = \pi_{\mathrm{span}(\theta)} f$.

The following theorem states that the projection of a high dimensional function onto a typical direction is fairly smooth.

Theorem 6.1. *If $f \in L^2(\mathbb{R}^d)$, $\mathrm{supp}(f) \subset B_1$, then provided that $\frac{d-1}{2} > \frac{1}{2} + k$, it holds*

$$\int_{\mathbb{S}^{d-1}} \|\pi_\theta f\|_{C^k}^2 \,\mathrm{d}\theta \lesssim \|f\|_{L^2}^2\,.$$

The following lemma builds a connection between the Fourier transform of f and that of its projection.

Lemma 6.2 (Dictionary)**.** *For any subspace $V \subset \mathbb{R}^d$ and any $\xi \in V$, $\widehat{\pi_V f}(\xi) = \hat{f}(\xi)$.*

Proof. By definition we have

$$\begin{aligned}\widehat{\pi_V f}(\xi) &= \int_V f(x) e^{-ix\cdot\xi}\,\mathrm{d}x = \int_V \int_{V^\perp} f(x+y)\,\mathrm{dvol}_{V^\perp}(y) e^{-ix\cdot\xi}\,\mathrm{d}x \\ &= \int_V \int_{V^\perp} f(x+y) e^{-i(x+y)\cdot\xi}\,\mathrm{dvol}_{V^\perp}(y)\,\mathrm{d}x \\ &= \int_{\mathbb{R}^d} f(x) e^{-ix\cdot\xi}\,\mathrm{d}x = \hat{f}(x)\,. \end{aligned}$$

□

Now we prove Theorem 6.1.

Proof. Using the dictionary lemma, we now relate $\|f\|_{L^2}$ with $\pi_\theta f$. By Plancherel's theorem and the polar coordinate transform, we have

$$\begin{aligned}\|f\|_{L^2}^2 = \|\hat{f}\|_{L^2}^2 &= \int_{\mathbb{R}^d} |\hat{f}(\xi)|^2 \,\mathrm{d}\xi \\ &= \int_{\mathbb{S}^{d-1}} \int_0^\infty |\hat{f}(r\theta)|^2 r^{d-1} \,\mathrm{d}r \,\mathrm{d}\theta \\ &= \frac{1}{2} \int_{\mathbb{S}^{d-1}} \int_{-\infty}^\infty |\widehat{\pi_\theta f}(r)|^2 r^{d-1} \,\mathrm{d}r \,\mathrm{d}\theta \,.\end{aligned}$$

Recall the Sobolev norms $\|\cdot\|_{\dot{H}_s}$ and $\|\cdot\|_{H_s}$: for $f : V \to \mathbb{C}$,

$$\|f\|_{\dot{H}^s}^2 = \int_V |\hat{f}(\xi)|^2 |\xi|^{2s} \,\mathrm{d}\xi\,, \quad \|f\|_{H^s}^2 = \int_V |\hat{f}(\xi)|^2 (1+|\xi|)^{2s} \,\mathrm{d}\xi\,.$$

The above estimate shows that

$$\int_{\mathbb{S}^{d-1}} \|\pi_\theta f\|_{\dot{H}^{\frac{d-1}{2}}}^2 \,\mathrm{d}\theta \lesssim \|f\|_{L^2}^2\,.$$

By the Sobolev embedding theorem, if $s > \frac{1}{2} + k$, then $\|\pi_\theta f\|_{C^k} \lesssim \|\pi_\theta f\|_{H^{\frac{d-1}{2}}}$. We have

$$\begin{aligned}\int_{\mathbb{S}^{d-1}} \|\pi_\theta f\|_{C^k}^2 \,\mathrm{d}\theta &\lesssim \int_{\mathbb{S}^{d-1}} \|\pi_\theta f\|_{H^{\frac{d-1}{2}}}^2 \,\mathrm{d}\theta \lesssim \int_{\mathbb{S}^{d-1}} \int_{-\infty}^\infty |\widehat{\pi_\theta f}(r)|^2 (1 + r^{d-1}) \,\mathrm{d}r \,\mathrm{d}\theta \\ &\lesssim \|f\|_{L^2}^2 + \int_{\mathbb{S}^{d-1}} \int_{|r|\le 1} |\hat{f}(r\theta)|^2 \,\mathrm{d}r \,\mathrm{d}\theta \lesssim \|f\|_{L^2}^2 + \|f\|_{L^1}^2\,.\end{aligned}$$

Finally, since f supports on B_1, we have $\|f\|_{L^1} \lesssim \|f\|_{L^2}$, the proof is completed. □

Connection to probability theory. Let $X_1, \ldots, X_N$ be independent random variables Uni($[-1/2, 1/2]$). The joint density of $X_1, \ldots, X_N$ is given by $f(\mathbf{x}) = \mathbf{1}_{[-1/2,1/2]^N}(\mathbf{x})$. According to central limit theorem,

$$\frac{1}{\sqrt{N}}(X_1 + X_2 + \cdots + X_N) \Rightarrow \mathcal{N}(0, 1/6)\,.$$

This suggests that the projection of f onto the direction $(\frac{1}{\sqrt{N}}, \ldots, \frac{1}{\sqrt{N}})$ approximates a Gaussian, which is a much smoother function than $\mathbf{1}_{[-1/2,1/2]}$.

Indeed, the central limit theorem says something more: provided the direction is not "close" to any coordinate axis, then the projection will approximate Gaussian. Similar results can be generalized from high dimensional cube to high dimensional convex bodies. This shows again the point that projecting onto a typical direction smoothens high dimensional functions.

18.156 Projection Theory
Spring 2025