8. The Szemeredi-Trotter theorem

Tues March 4

The Szemeredi-Trotter theorem gives the sharp answer to a natural discrete projection problem in the plane. It was proven in the early 1980s. The proof of the theorem is based on topology, and it is completely different from the proofs we have explored earlier. Tom Wolff noticed the connection between the Szemeredi-Trotter theorem and problems in geometric measure theory like the exceptional set problem and the Furstenberg set problem.

8.1. The Szemeredi-Trotter projection theorem.

Theorem 8.1. *Let X be a set of points in $\mathbb{R}^2$ and D a set of directions in S^1. Then we define*

$$S(X, D) = \max_{\theta \in D} |\pi_\theta(X)| \tag{20}$$

Then

$$|D| \leq \frac{S^2}{|X|} + 1 \tag{21}$$

Now for the general theorem, let X be a set of points in $\mathbb{R}^2$ and L a set of lines in $\mathbb{R}^2$. Then we define

$$I(X, L) := \#\{x \in X, \ell \in L, x \in \ell\}$$

Note that

$$I(X, L) = \sum_{\ell \in L} |\ell \cap X|$$

Then the SzemerdiTrotter (ST) theorem states that

Theorem 8.2.

$$I(X, L) \leq |X| + |L| + |X|^{2/3}|L|^{2/3} \tag{22}$$

Example 8.3 (Example 1 for ST Theorem)**.** *The ST theorem is sharp with the $|X|$ bound when the number of lines is small and each point lies on a single line.*

Example 8.4 (Example 2 for ST Theorem)**.** *The ST theorem is sharp with the $|L|$ bound when the number of lines is large and each line lies on a single point.*

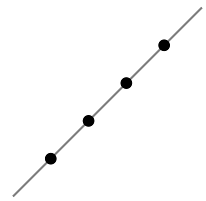

FIGURE 10. Example of setup where $|X|$ term dominates and $I \sim |X|$

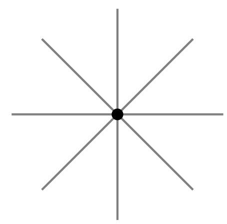

FIGURE 11. Example of setup where $|L|$ term dominates and $I \sim |L|$.

Example 8.5 (Example 3 for ST Theorem)**.** *We let X be an $N \times N$ grid, and define $Q_M := \{\frac{a}{b} : a, b \in [M]\}$. Define L to be the set of lines with slopes in Q_M that pass through points in X. Then $|Q_M| \sim M^2$ (the double counting when* $\gcd(a,b) > 1$ *only affects the magnitude of $|Q_M|$ up to a constant factor). Every point in X has a line passing through it for each slope in Q_M. Then*

$$I(X,L) = |X||Q_M| \sim N^2M^2 \tag{23}$$

We now define projection operators for each $s \in Q_M$ as

$$\pi_s(x_1,x_2) = x_2 = -sx_1 \tag{24}$$

The fibers of π_s are lines of slope s, and the number of lines in L with direction s is $|\pi_s(X)|$. We now prove the following lemma:

Lemma 8.6. *For all $s \in Q_M$, $|\pi_s(X)| \lesssim MN$*

Proof. Take $x_1, x_2 \in [N]$ and $s = \frac{a}{b}$. Then

$$\pi_s(x_1,x_2) = x_2 - \frac{a}{b}x_2 = \frac{bx_2 - ax_1}{b} \tag{25}$$

Since $a, b \leq M$ and $x_1, x_2 \leq N$, $|bx_2 - ax_1| \lesssim MN$. Since $bx_2 - ax_1$ must be an integer, there are at most MN distinct values in $\pi_s(X)$. $\square$

Then since L has at most NM lines for every element of Q_M. $|L \lesssim |Q_M|NM \sim M^3N$. Then

$$I(X,L) \sim N^2M^2 \gtrsim (M^3N \cdot N^2)^{2/3} \gtrsim |X|^{2/3}|L|^{2/3} \tag{26}$$

Therefore the grid is a sharp example of the SzemerdiTrotter theorem where the $|X|^{2/3}|L|^{2/3}$ term dominates. Note that the SzemerdiTrotter theorem implies the ST projection theorem, which is a special case when L is the set of lines with directions in D passing through points in X.

8.2. **Question: Are there other sharp examples for the SzemerdiTrotter theorem?** Another example is grids over number fields. Let R be a number field, (for example $\mathbb{Z}[\sqrt{2}]$). Then define

$$R_N = \{a_1 + a_2\sqrt{2} : a_1, a_2 \in \mathbb{Z}, |a_1|, |a_2| \leq N\}$$

$$QR_M = \{\frac{a}{b} : a, b \in R_M\}$$

Then define $X := R_N \times R_N$ and L as the set of lines with slopes in QR_M that pass through a point in X. This is similar to the grid example.

8.3. **Proof of the Szemeredi-Trotter theorem.** We begin the proof of the Szemeredi-Trotter theorem with a lemma.

Lemma 8.7.

$$I(X,L) \leq |X||L|^{1/2} + |L|$$

Proof. We start with expanding $I(X,L)$ and applying Cauchy Schwartz to get

$$I(X,L) = \sum_{\ell \in l} |\ell \cap X| \leq \left(|L| \sum_{\ell \in L} |\ell \cap X|^2\right)^{1/2}$$

This is advantageous because $|\ell \cap X|^2 \lesssim \binom{|\ell \cap X|}{2} + 1$. Then

$$\sum_{\ell \in L} |\ell \cap X|^2 \lesssim |L| + \sum_{\ell \in L} \binom{|\ell \cap X|}{2}$$

Since for every pair of points $x_1, x_2 \in X$, there is at most one line ℓ that contains x_1 and x_2, every pair of points in X can be counted at most once. Then

$$\sum_{\ell \in L} \binom{|\ell \cap X|}{2} \leq \binom{|X|}{2} \lesssim |X|^2$$

This gives the final conclusion

$$I(X,L) \lesssim \left(|L|(|X|^2+|L|)\right)^{1/2} \leq |X||L|^{1/2}+|L|$$

□

Note that this proof uses only the very general fact that any two points define a line. Therefore it holds over spaces such as finite fields. However, the SzemerdiTrotter theorem does not hold over finite fields. To see this take $X=\mathbb{F}_q^2$ (as the whole space) and L as all lines in $\mathbb{F}_q^2$. Then for every ℓ, $|\ell \cap X| = q$, so $I(X,L)=q^3$. However, $|X|^{2/3}|L|^{2/3} = q^{8/3} \leq I(X,L)$. Therefore, the SzemerdiTrotter theorem requires properties of the topology of $\mathbb{R}^2$ to work. In particular, it uses a cell decomposition lemma, which allows cutting the plane into pieces.

Lemma 8.8 (Cell decomposition lemma)**.** *Let X be a set of points in $\mathbb{R}^2$ and pick an integer $s \geq 1$. Then the plane can be disjointly partitioned into a set of open sets O_i and a closed set W such that*

$$\mathbb{R}^2 = W \cup \bigcup_i O_i$$

and additionally,

$$|\ell \cap W| \leq s$$

and for every i,

$$|X| \cap O_i| \lesssim \frac{|X|}{s^2}$$

This lemma essentially states that the plane can be split into cells that each contain only a small subset of X, and that the walls don't intersect any line too many times. As an example of this theorem, let X be a "roughly" square grid. That is $\subset [N]^2$ and fir every ball $B_1(c)$ of radius 1 (where c is an arbitrary point in the plane), $|X \cap B_1(c)| \lesssim 1$. The below example shows the grid for $s=2$.

Each line can only intersect $2s$ lines in W, so $|\ell \cap W| \leq 2s$. Since X is roughly grid shaped, and each cell is a square of side length N/s, $|X \cap O_i| \lesssim |X|/s^2$, which satisfies the requirements.

We now proceed to the proof of the SzemerdiTrotter theorem. It hinges on the fact that lemma 8.7 is sharp when the number of lines is either small (bounded by a constant) or much larger than the number of points ($|L| > |X|^2$). We can use this by using the cell decomposition lemma to pick cells where one of these conditions holds.

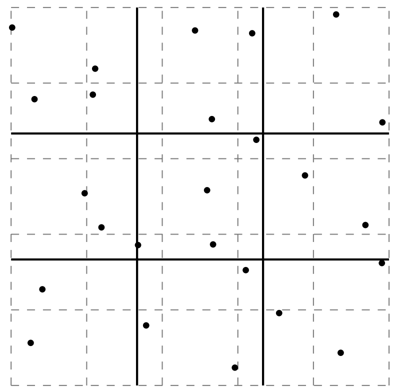

FIGURE 12. The points show the points in X. The dashed lines indicate the "rough grid" shape of X. The solid lines show W, which is the set dividing X

Proof. Given X and L, and arbitrary s. Then using the cell decomposition lemma, define $X_i = X \cap O_i$. Then $|X_i| \leq |X|/s^2$ and

$$\sum_i |X_i| \leq |X|$$

Define

$$L_i = \{\ell \in L : \ell \cap O_i \neq \emptyset\}$$

From the cell decomposition lemma

$$\sum_i |L_i| \leq s|L|$$

Then as every intersection of a point and a line is either on a cell boundary or within a cell. Then

$$I(X, L) \leq \sum_i I(X_i, L_i) + I(X \cap W, L)$$

Applying lemma 8.7 to the first term, and the $|\ell \cap W| \lesssim s$ bound to the second term,

$$\begin{aligned}
I(X,L) &\lesssim \sum_i (|X_i||L_i|^{1/2} + |L_i|) + s|L| \\
&\lesssim \left(\sum_i |X_i|^2 \sum_i |L_i| \right)^{1/2} + 2s|L| \\
&\lesssim \left(\sum_i \frac{|X|}{s^2} |X_i| \right)^{1/2} (s|L|)^{1/2} + 2s|L| \\
&\lesssim s^{-1/2}|X||L|^{1/2} + s|L|
\end{aligned}$$

We then choose s to minimize this quantity. This is effectively choosing s so that both terms are equal. Then

$$\begin{aligned}
s^{-1/2}|X||L|^{1/2} =& s|L| \\
|X||L|^{-1/2} =& s^{3/2} \\
s =& |X|^{2/3}|L|^{-1/3}
\end{aligned}$$

Plugging s back into the inequality gives

$$I(X,L) \lesssim |X|^{2/3}|L|^{2/3}$$

which gives the desired bound. □

Note that in the above argument s must be an integer, so this can only be done when $|X|^2 > |L|$. When $|X|^2 < |L|$ then setting $s = 1$ gives the the bound $I(X,L) \lesssim |L|$. Additionally, s^2 can be at most $|X|$. Then when $|X| < |X|^{4/3}|L|^{-2/3}$, $|X| > |L|^2$, so setting $s = |X|^{1/2}$ gives the bound $I(X,L) \lesssim |X|$

We now prove the cell decomposition lemma. However, several prelimary theorems must be shown first.

Theorem 8.9 (Borsuk Ulam Theorem)**.** *Let $f : S^n \to \mathbb{R}^n$ be a continuous function that is antipodal, ie for every $\theta \in S^n$, $f(\theta) = -f(-\theta)$. Then 0 is in the image of f.*

Corollary 8.10 (Ham Sandwich Theorem)**.** *Let $O_1, O_2, ..., O_n \subset \mathbb{R}^n$ be bounded open subsets. Then there exists a hyperplane H that bisects every O_i.*

Proof. An upper half (hyper)plane can be described as the set $\{x : a \cdot x > b\}$, for some vector a and b a real number. As scaling a and b by a positive real number preserves this hyperplane, the tuple (a, b), can be identified with an element of S^n.

Then for an element $\theta \in S_n$ corresponding to (a, b), we let c_θ be the affine operator defined by $c_\theta(x) = a \cdot x + b$. We then define the vector valued function f by

$$f_i(\theta) = \mathrm{Vol}(O_i \cap \{x : c_\theta(x) > 0\}) - \mathrm{Vol}(O_i \cap \{x : c_\theta(x) < 0\})$$

f is antipodal, so it has a zero. This zero corresponds to each set being bisected, which proves the theorem □

The Ham Sandwich theorem works when there are up to n subsets of $\mathbb{R}^n$. This is roughly because n degrees of freedom are needed to bisect the n sets. We can then use polynomials to increase the number of degrees of freedom, and so the number of sets that can be bisected.

Theorem 8.11 (Polynomial Ham Sandwich Theorem)**.** *We use the same setup as the ham sandwich theorem, except that there can be up to N sets O_i. Then there exists a polynomial zero set that bisects every O_i*

Proof. First define the space

$$\mathrm{Poly}_D(\mathbb{R}^n) = \{p \in \mathbb{R}[x_1, ..., x_n] : \deg p \leq D\}$$

$\mathrm{Poly}_D(\mathbb{R}^n)$ is then a vector space of degree D^n. We claim that if $N < D^n$, then there is a nonzero element of $\mathrm{Poly}_D(\mathbb{R}^n)$ that satisfies the claim. We define the vector valued function f by

$$f_i(p) = \mathrm{Vol}(O_i \cap \{x : p(x) > 0\}) - \mathrm{Vol}(O_i \cap \{x : p(x) < 0\})$$

Since scaling each nonzero p by a positive real does not change f, f is a function from S^{D^N-1} to $\mathbb{R}^n$. Additionally, f is antipodal. Then by the Borsuk Ulam theorem the conclusion follows. □

The Ham Sandwich theorems allow open subsets of Euclidean space to be subdivided, but the cell decomposition lemma requires dividing sets of points. This is a technical detail that follows from the Polynomial Ham Sandwich Theorem.

Lemma 8.12 (Ham Sandwich theorem for finite sets)**.** *Let $s_1, s_2, ..., s_N$ be a set of finite sets in $\mathbb{R}^n$. Then there exists a polynomial level set such that for every s_i,*

$$|s_i \cap \{x : p(x) > 0\}| \leq \frac{|s_i|}{2}$$
$$|s_i \cap \{x : p(x) < 0\}| \leq \frac{|s_i|}{2}$$

As individual points cannot be bisected, this lemma instead guarantees that excess points will lie on the level set.

Proof. Take some $\epsilon > 0$ and for each i define $N_\epsilon(s_i)$ to be the set of balls of radius ϵ centered at the points on s_i. Then by the Polynomial Ham Sandwich theorem the $N_\epsilon(s_i)$ can all bisected by the zero set of a polynomial of degree $D^n > n$. Then taking ϵ to 0 we get a sequence of polynomials p_ϵ that each bisect the $N_\epsilon(s_i)$. Since the sphere S^n is compact, there must a convergent subsequence to some polynomial p. To show that this polynomial p satisfies the conclusion, for contradiction assume that there exists i such that

$$|s_i \cap \{x : p(x) > 0\}| > \frac{1}{2}|s_i|$$

Since every point in $s_i \cap \{x : p(x) > 0\}$ is some nonzero distance from the set $\{x : p(x) = 0\}$, there is some $\epsilon > 0$ such that modifying p by ϵ and enlarging s_i by ϵ gives

$$|N_\epsilon(s_i) \cap \{x : p_\epsilon(x) > 0\}| > \frac{1}{2}|N_\epsilon(s_i)| =$$

a contradiction. Note that this step requires the boundedness of s_i to take a perturbation of p continuous. □

We will now prove the cell decomposition lemma.

Proof. We begin with step $k = 1$. Then define p_1 to be the degree 1 polynomial that splits X into two parts. Then $X_{1,1} := \{x \in X : p_1(x) > 0\}$ and $X_{1,2} := \{x \in X : p_1(x) < 0\}$

Then at stek $k+1$, define p_k to be the polynomial of degree D_k with $D^2 \sim 2^k$ such that p_k bisects all $X_{k,1}, ..., X_{k,2^k}$. Then $D_k \sim 2^{k/2}$.

Then pick k_{final} such that $2^{k_{final}} \sim s^2$. Then let O_i be the sets defined by $\{x : \pm p_1(x) > 0\} \cap ... \cap \{x : \pm p_{k_{final}}(x) > 0\}$ for all choices of $\pm$. Define $W = \{x : p_1(x) = 0\} \cup ... \cup \{x : p_{k_{final}}(x) = 0\}$

X has been bisected k_{final} times, so then

$$|X_{k_{final},i}| \lesssim |X|/s^2$$

A line can intersect a polynomial of degree D at most D times, so then

$$|\ell \cap W| \leq 1 + 2 + 4 + ... + 2^{k_{final}/2} \sim s$$

Then each line can intersect W at most $\sim s$ times. □

18.156 Projection Theory
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